



Bornes sur des valeurs propres et métriques extrémales

Romain Petrides

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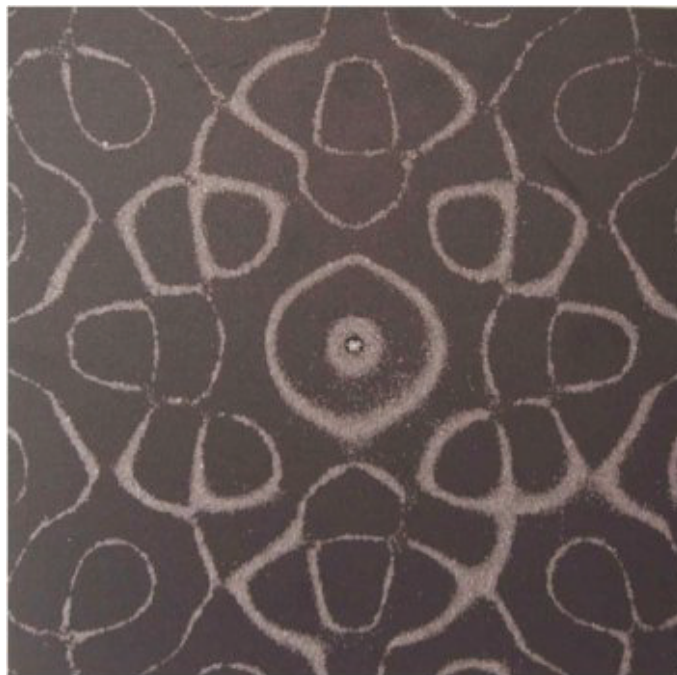
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Bornes sur des valeurs propres et métriques extrémales



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Thèse de doctorat

Université Claude Bernard Lyon 1
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Mis en page avec la classe thesul.

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Introduction

1 Bornes sur les valeurs propres de Laplace

1.1 Motivations physiques et historiques

Dans cette thèse, nous nous attacherons à étudier les fréquences propres émises par une membrane rigide en fonction de la forme géométrique de sa surface. Ce lien géométrique a été l'un des plus étudiés en analyse fonctionnelle et en géométrie différentielle depuis plus d'un siècle. Il est à l'origine de méthodes très variées dans ces domaines et de manière surprenante, comme on le verra dans la section 2, donne des applications à la théorie des surfaces minimales.

Partons des origines du domaine : un ouvrage de Lord Rayleigh, [96], *Theory of sound*, 1894. Dans cet écrit, il conjecture grâce à des calculs et des constatations physiques simples que *la fréquence fondamentale émise par un tambour est toujours plus aigüe que celle émise par un tambour circulaire de même aire*.

Voici la modélisation physique du problème. On entend ici par tambour une membrane rigide dont la surface est fixée à son bord. Penser ici à l'instrument de musique ! Ce bord est une courbe fermée du plan qui est la frontière d'un ouvert Ω de \mathbb{R}^2 . On peut alors paramétrer un point $(x, y, z) \in \mathbb{R}^3$ de la membrane par un point $(x, y) \in \Omega$ via la fonction hauteur $z(x, y)$. Des forces s'exercent sur la membrane, parce qu'elle est tendue. On note τ la tension superficielle : c'est la force par unité de longueur normale à une ligne de la membrane. On note μ la masse par unité de surface, constante car on suppose la membrane homogène. On note aussi $\Delta = -(\partial_x^2 + \partial_y^2)$ le Laplacien géométrique en un point $z = (x, y)$ du plan \mathbb{R}^2 . Le signe $-$ est une convention qu'utilisent les géomètres pour avoir un opérateur positif. Alors les forces verticales qui s'exercent sur un élément de surface $dx dy$ en un point $(x, y) \in \Omega$ sont

- La force de tension : $-\tau \Delta z dx dy$
- Le poids : $-\mu g dx dy$

La deuxième loi de Newton s'écrit alors

$$\mu \partial_t^2 z dx dy = -\tau \Delta z dx dy - \mu g dx dy .$$

Au repos, la membrane fléchit sous son propre poids et sa hauteur $z_0(x, y)$ vérifie l'équation $\tau \Delta z_0 + \mu g = 0$. L'écart au repos $\Psi = z - z_0$ vérifie alors l'équation d'onde

$$\partial_t^2 \Psi + \frac{\tau}{\mu} \Delta \Psi = 0 \text{ sur } \Omega \times \mathbb{R} .$$

En cherchant les modes propres de la forme $\Psi(z, t) = \phi(z) e^{i\omega t}$, où $\phi : \Omega \rightarrow \mathbb{R}$ est l'amplitude

de l'onde sur la surface et ω une fréquence propre, ϕ satisfait

$$\begin{cases} \Delta\phi = \frac{\omega^2\mu}{\tau}\phi & \Omega \\ \phi = 0 & \partial\Omega. \end{cases} \quad (1)$$

La première condition vient directement de l'équation d'onde alors que la seconde est une condition aux limites venant du fait que la membrane est laissée fixe au bord. On obtient ici l'équation aux valeurs propres de Laplace avec condition de Dirichlet au bord. Pour avoir une solution, une valeur propre du Laplacien λ impose une condition de compatibilité sur la fréquence $\lambda = \frac{\omega^2\mu}{\tau}$ et ϕ doit être une fonction propre associée.

Si Ω est borné, grâce à des résultats classiques de diagonalisation du Laplacien, les solutions (λ, ϕ) viennent d'un spectre discret

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_k(\Omega) \rightarrow +\infty$$

de valeurs propres et d'une base hilbertienne dans $L^2(\Omega)$

$$\phi_1, \phi_2, \dots, \phi_k, \dots$$

de fonctions propres associées qui sont régulières. La résolution de l'équation (1) est équivalente à la donnée de ces valeurs propres et fonctions propres qui ne dépendent que de la forme et de la taille de Ω .

La conjecture de Rayleigh porte sur $\lambda_1(\Omega)$, la plus petite valeur propre non nulle du Laplacien, qui correspond à la fréquence fondamentale du tambour. Cette fonctionnelle dépend de la forme Ω du tambour et de sa taille. Par un changement de variable $\tilde{\phi}(z) = \phi_1(\frac{z}{\alpha})$, on obtient une solution de l'équation (1) sur $\alpha\Omega$ avec

$$\lambda_1(\alpha\Omega) = \frac{1}{\alpha^2} \lambda_1(\Omega).$$

Afin de n'étudier que l'influence de sa forme géométrique, il suffit donc de fixer l'aire de notre surface ou d'étudier la fonctionnelle $A(\Omega)\lambda_1(\Omega)$, invariante par dilatation (où $A(\Omega)$ est l'aire de Ω). La conjecture de Rayleigh s'écrit alors

$$A(\Omega)\lambda_1(\Omega) \geq \pi\lambda_1(\mathbb{D}),$$

où \mathbb{D} est le disque unité de \mathbb{R}^2 .

Plus généralement, cette question a été posée en dimension supérieure pour des ouverts de \mathbb{R}^n . La réponse devient

$$V(\Omega)^{\frac{2}{n}} \lambda_1(\Omega) \geq \omega_n^{\frac{2}{n}} \lambda_1(\mathbb{B}^n),$$

où \mathbb{B}^n est la boule unité de \mathbb{R}^n , $V(\Omega)$ le volume de Ω et $\omega_n = V(\mathbb{B}_n)$ le volume de la boule unité. Cette inégalité porte le nom d'inégalité de Faber-Krahn du nom de ceux qui l'ont démontrée indépendamment en 1923 [37], [68]. De plus, cette inégalité est une égalité si et seulement si Ω est une boule.

Noter que dans ce résultat, on a à la fois l'existence d'une borne inférieure sur la fonctionnelle $A(\Omega)\lambda_1(\Omega)$ et le fait que cette borne est atteinte par une surface dite extrémale. Ces deux questions nous intéresseront particulièrement dans la suite. On dit qu'on a un problème

d'optimisation de formes. Les problèmes d'optimisation de formes du même type que Faber-Krahn peuvent être posés sur d'autres fonctionnelles comme les valeurs propres suivantes $A(\Omega)\lambda_k(\Omega)$ ou des fonctions dépendant de ces valeurs propres $F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$. Noter aussi que pour Faber-Krahn, la surface extrémale est explicite, ce qui n'est presque jamais le cas.

On peut aussi changer la nature physique du problème et s'intéresser au spectre de Neumann. Dans ce cas, on a la même équation physique, seule la condition aux limites change : on laisse la surface vibrer librement à son bord. On trouve par le même procédé que pour le tambour les solutions (μ, ϕ) de

$$\begin{cases} \Delta\phi = \mu\phi & \Omega \\ \partial_\nu\phi = 0 & \partial\Omega. \end{cases} \quad (2)$$

où ∂_ν est la dérivée normale sur le bord dont le spectre est noté :

$$0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots \leq \mu_k(\Omega) \rightarrow +\infty,$$

où en supposant Ω connexe, la première valeur propre $\mu_0 = 0$ est simple et est associée aux fonctions constantes.

Au cours de la thèse, on va surtout étudier le spectre du Laplacien $\Delta_g = -\text{div}_g \nabla$ des variétés Riemanniennes connexes compactes sans bord (M, g) noté

$$0 = \lambda_0 < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \dots \leq \lambda_k(M, g) \rightarrow +\infty,$$

où ici, la forme géométrique de la variété M est déterminée par la métrique Riemannienne g qu'on lui attribue. De même, la première valeur propre $\lambda_0 = 0$ est simple et associée aux fonctions constantes.

D'autres questions sur le spectre, vu comme la suite des valeurs propres, peuvent être posées. Par exemple Weyl donne une estimation asymptotique sur la suite des valeurs propres du Laplacien avec conditions de Dirichlet en 1911

$$\lambda_k(\Omega) \sim c_n k^{\frac{2}{n}} \quad (3)$$

lorsque k tend vers $+\infty$, où Ω est un domaine de \mathbb{R}^n de volume 1 de dimension n fixée et

$$c_n = \frac{(2\pi)^2}{\omega_n^{\frac{2}{n}}}$$

ne dépend que de la dimension. Des estimations asymptotiques analogues peuvent être données sur tous les spectres définis précédemment. Un autre exemple est la célèbre question de Marc Kac en 1966 [58] : *Can one hear the shape of a drum?* Autrement dit, deux surfaces non isométriques ont-elles forcément un spectre différent? La réponse à cette question est non et ce n'est qu'en 1992 que deux surfaces non isométriques de même spectre de Dirichlet sont exhibées [48]. Nous ne traiterons pas ces types de questions au cours de la thèse, mais nous nous intéresserons plutôt à des problèmes d'inégalité et d'optimisation de formes sur une valeur propre donnée : les questions de type Faber-Krahn.

1.2 Optimisation et domaines extrémaux dans le cadre classique

Pour toute cette section, nous renvoyons en complément à la référence [104], chapitre III. Un premier objectif de la thèse est de donner des bornes sur une valeur propre $\lambda_k(M, g)$ d'une variété riemannienne (M, g) en fonction de la forme géométrique de la variété donnée par sa métrique g , sous certaines contraintes. Le deuxième est de démontrer l'existence ou la non-existence de variétés pour lesquels ces bornes sont atteintes. Afin de bien cerner ces questions, nous proposons en plus de dresser un état de l'art sur les valeurs propres du Laplacien sur un domaine Ω de \mathbb{R}^n avec conditions de Dirichlet $\lambda_k(\Omega)$ et des valeurs propres du Laplacien avec conditions de Neumann $\mu_k(\Omega)$.

Pour obtenir des inégalités sur $\lambda_k(M, g)$, nous utiliserons la caractérisation variationnelle de la k -ième valeur propre du Laplacien sur une variété riemannienne compacte (M, g) :

$$\lambda_k(M, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_M \phi^2 dv_g}, \quad (4)$$

où l'infimum est pris parmi les espaces vectoriels de fonctions \mathcal{C}^∞ de dimension $k + 1$. Noter ici que cet infimum est atteint si E_{k+1} est la somme des espaces propres associés aux valeurs propres inférieures à λ_k et le supremum pour une fonction propre associée à λ_k dans cet espace. On appelle quotient de Rayleigh le quotient qui apparaît dans le supremum et énergie de Dirichlet son numérateur. Cette caractérisation existe aussi pour $\lambda_k(\Omega)$ et $\mu_k(\Omega)$ lorsque Ω est un domaine de \mathbb{R}^n .

Remarquons d'abord que certaines bornes sont triviales, même à volume fixé :

Proposition 1. *En prenant l'infimum et le suprémum suivants sur les ouverts connexes bornés Ω de \mathbb{R}^n , on obtient :*

$$\begin{aligned} \sup_{\Omega} V(\Omega)^{\frac{2}{n}} \lambda_k(\Omega) &= +\infty \\ \inf_{\Omega} V(\Omega)^{\frac{2}{n}} \mu_k(\Omega) &= 0 \end{aligned}$$

et sur toute variété connexe compacte sans bord :

$$\inf_g \text{Vol}_g(M)^{\frac{2}{n}} \lambda_k(M, g) = 0,$$

où $\text{Vol}_g(M)$ est le volume de la variété riemannienne (M, g) .

Démonstration.

Pour les domaines de \mathbb{R}^n , une suite optimisante de formes sont des parallélépipèdes rectangles dont une longueur est très grande par rapport aux autres :

$$[0, L] \times [0, 1]^{n-1} \text{ avec } L \rightarrow +\infty,$$

pour lesquels les valeurs propres se calculent explicitement dans les deux cas λ_k et μ_k et vérifient l'asymptotique voulue.

Pour les variétés riemanniennes compactes, il suffit de choisir une suite de métriques $\{g_m\}_{m \geq 1}$ satisfaisant

$$g_m = e^{2u_m} g \text{ et } \int_M e^{nu_m} dv_g = 1$$

de sorte que la forme volume associée converge au sens des mesures (pour la topologie faible \star) vers une somme de $k + 1$ masses de Dirac :

$$dv_{g_m} = e^{nu_m} dv_g \rightharpoonup_{\star} \frac{\delta_{x_1} + \cdots + \delta_{x_{k+1}}}{k + 1},$$

où x_1, \dots, x_{k+1} sont $k + 1$ points distincts de M . Ici, g est une métrique de référence fixée arbitrairement.

Pour des raisons de capacité, pour $r > 0$ fixé, on peut trouver des fonctions de classe \mathcal{C}^∞ , $\phi_1, \dots, \phi_{k+1}$ telles que pour $1 \leq i \leq k + 1$, $0 \leq \phi_i \leq 1$, $\phi_i = 1$ dans $B_g(x_i, r)$, $\phi_i = 0$ dans $M \setminus B_g(x_i, \sqrt{r})$ et

$$\int_M |\nabla \phi_i|_g^n dv_g \leq \frac{C}{\ln\left(\frac{1}{r}\right)},$$

où C est indépendante de r . Tester ces fonctions à supports disjoints dans la caractérisation min-max (4) donne

$$\begin{aligned} \lambda_k(M, g_m) &\leq \sup_{\phi \in \langle \phi_1, \dots, \phi_{k+1} \rangle \setminus \{0\}} \frac{\int_M |\nabla \phi|_{g_m}^2 dv_{g_m}}{\int_M \phi^2 dv_{g_m}} \\ &\leq \max_{1 \leq i \leq k+1} \frac{\int_M |\nabla \phi_i|_g^2 e^{(n-2)u_m} dv_g}{\int_M \phi_i^2 e^{nu_m} dv_g} \\ &\leq \max_{1 \leq i \leq k+1} \frac{\left(\int_M |\nabla \phi_i|_g^2 dv_g \right)^{\frac{2}{n}} \left(\int_M e^{nu_m} dv_g \right)^{\frac{n-2}{n}}}{\int_M \phi_i^2 e^{nu_m} dv_g} \\ &\leq \frac{C^{\frac{2}{n}}}{\ln\left(\frac{1}{r}\right)^{\frac{2}{n}}} \max_{1 \leq i \leq k+1} \frac{1}{\frac{1}{k+1} + \int_{B_g(x_i, \sqrt{r})} \phi_i^2 \left(e^{nu_m} dv_g - \frac{\delta_{x_i}}{k+1} \right)} \\ &\rightarrow \frac{(k+1)C^{\frac{2}{n}}}{\ln\left(\frac{1}{r}\right)^{\frac{2}{n}}} \text{ lorsque } m \rightarrow +\infty. \end{aligned}$$

En faisant tendre r vers 0, on obtient le résultat. ◇

Cette proposition nous dit que les seules bornes pertinentes à étudier sont la borne inférieure pour le spectre de Dirichlet dans des domaines de \mathbb{R}^n et la borne supérieure pour le spectre de Neumann dans des domaines de \mathbb{R}^n ou pour le spectre des variétés compactes sans bord, les autres étant triviales.

Donnons successivement les principaux résultats concernant les première et deuxième valeurs propres non nulles dans les cas Dirichlet et Neumann pour des domaines de \mathbb{R}^n et pour la sphère de dimension 2. Ils donnent un bel aperçu des problématiques et révèlent les questions qui se posent sur le sujet.

Enonçons d'abord l'inégalité de Faber-Krahn démontrée pour la première valeur propre avec condition de Dirichlet :

Théorème 1 (Faber [37], Krahn [68]). *Pour tout domaine $\Omega \subset \mathbb{R}^n$, on a*

$$\lambda_1(\Omega) V(\Omega)^{\frac{2}{n}} \geq \lambda_1(\mathbb{B}^n) \omega_n^{\frac{2}{n}}$$

avec égalité si et seulement si Ω est une boule.

La démonstration utilise des techniques de réarrangements symétriques. On obtient une conséquence quasiment immédiate pour la deuxième valeur propre :

Théorème 2 (Krahn [69]). *Pour tout domaine $\Omega \subset \mathbb{R}^n$, on a*

$$\lambda_2(\Omega) V(\Omega)^{\frac{2}{n}} \geq \lambda_1(\mathbb{B}^n) (2\omega_n)^{\frac{2}{n}}$$

avec égalité si et seulement si Ω est une union disjointe de deux boules de même volume.

Le résultat est même plus précis en dimension 2 pour les domaines connexes du plan. En effet dans ce cas, l'inégalité est optimale. Par conséquent, il n'existe pas d'ouvert connexe minimisant la deuxième valeur propre.

S'agissant du spectre de Neumann, il existe un résultat analogue à l'inégalité de Faber-Krahn démontré par Weinberger :

Théorème 3 (Weinberger [111]). *Pour tout domaine $\Omega \subset \mathbb{R}^n$, on a*

$$\mu_1(\Omega) V(\Omega)^{\frac{2}{n}} \leq \mu_1(\mathbb{B}^n) \omega_n^{\frac{2}{n}}$$

avec égalité si et seulement si Ω est une boule.

Ce résultat avait déjà été démontré par Szegö en dimension 2 [107] pour des domaines simplement connexes. Les preuves de ce résultat sont rassemblées dans [104]. Un résultat analogue au cas Dirichlet pour la deuxième valeur propre non nulle a été démontré récemment pour les domaines simplement connexes du plan par Girouard, Nadirashvili et Polterovich.

Théorème 4 (Girouard, Nadirashvili, Polterovich [46]). *Pour tout domaine Ω simplement connexe du plan, on a*

$$\mu_2(\Omega) A(\Omega) \leq 2\pi \mu_1(\mathbb{D})$$

où \mathbb{D} est le disque unité de \mathbb{R}^2 . De plus l'inégalité est optimale où le cas d'égalité, qui est dégénéré, est l'union disjointe de deux disques de même aire.

Pour les sphères de dimension 2, on a le résultat de Hersch pour la première valeur propre non nulle :

Théorème 5 (Hersch [54]). *Soit g une métrique quelconque de \mathbb{S}^2 , alors*

$$\lambda_1(\mathbb{S}^2, g) A_g(\mathbb{S}^2) \leq 8\pi = \lambda_1(\mathbb{S}^2, g_0) A_{g_0}(\mathbb{S}^2)$$

où g_0 est la métrique standard de la sphère ronde. On a égalité si et seulement si g est une métrique ronde.

Enfin, on peut énoncer un résultat de Nadirashvili et simplifié par l'auteur de la thèse (voir le chapitre 1) sur la deuxième valeur propre non nulle sur les sphères :

Théorème 6 (Nadirashvili [84], P. [92]). *Soit g une métrique quelconque de \mathbb{S}^2 , alors*

$$\lambda_2(\mathbb{S}^2, g) A_g(\mathbb{S}^2) < 16\pi = 2\lambda_1(\mathbb{S}^2, g_0) A_{g_0}(\mathbb{S}^2)$$

et l'inégalité est optimale où le cas d'égalité, qui est dégénéré, est l'union disjointe de deux sphères de même aire.

On remarque dans les théorèmes 1, 3 et 5 qu'optimiser la première valeur propre non nulle a tendance à uniformiser la forme extrême (on obtient des boules et des sphères rondes). En optimisant les valeurs propres, on obtient donc des formes géométriques très particulières. Il en sera de même pour les exemples sur les surfaces riemanniennes dans les prochaines sections.

De la preuve de la proposition 1 et des théorèmes 2, 4 et 6, on tire cette idée qu'on obtient certaines formes extrêmes en "déconnectant" la surface. Pour obtenir la deuxième valeur propre non nulle optimale, la forme extrême a tendance à être une union disjointe de deux formes correspondant à la forme extrême pour la première valeur propre (des boules et des sphères).

Intéressons-nous à la k -ième valeur propre pour $k \geq 3$ dans le cas Dirichlet et Neumann. A ce jour, on ne connaît pas les formes extrêmes associées, même en dimension 2. On peut naturellement penser que l'union disjointe de k boules de même volume est extrême, mais ceci est faux par la loi de Weyl (3) qui est aussi vraie pour le spectre de Neumann. En dimension 2, il est conjecturé que c'est à nouveau le disque qui minimise λ_3 , comme l'attestent des calculs approchés [87]. Des simulations numériques donnent aussi des formes optimales pour les petits rangs de λ_k et μ_k [87], [2]. Il est même démontré que dans le cas Dirichlet, à partir de $k = 4$ [5], les formes extrêmes ne sont jamais des disques et qu'à partir de $k = 5$, elles ne sont pas toujours des unions de disques [87].

Cependant, on sait depuis récemment qu'il existe toujours des minimiseurs (voir Bucur [12] et Mazzoleni-Pratelli [78]) pour les valeurs propres avec conditions de Dirichlet $\lambda_k(\Omega)$.

Ne connaissant pas les formes optimales, on peut tout de même conjecturer des bornes sur les valeurs optimales. La conjecture de Pólya dit que

$$V(\Omega)^{\frac{2}{n}} \lambda_k(\Omega) \geq c_n k^{\frac{2}{n}},$$

$$V(\Omega)^{\frac{2}{n}} \mu_k(\Omega) \leq c_n k^{\frac{2}{n}},$$

pour tous n, k , et Ω domaine de \mathbb{R}^n , où c_n est la même constante que celle donnée par la loi de Weyl (3). En particulier, $c_2 = 4\pi$.

Dans le cas de la sphère de dimension 2, l'analogue de la conjecture de Pólya est

$$\lambda_k(\mathbb{S}^2, g) A_g(\mathbb{S}^2) \leq 8\pi k,$$

où $8\pi k$ correspond au cas dégénéré de k sphères rondes disjointes de même aire. Si cette inégalité se présentait, la situation serait totalement différente de la conjecture de Pólya pour les domaines de \mathbb{R}^n car elle serait optimale avec pour cas d'égalité k formes de même aire associées à la première valeur propre. Comme on le verra, cette particularité vient d'une géométrie plus riche dans le cas des variétés riemanniennes pour les maximiseurs que dans le cadre classique.

1.3 Optimisation et domaines extrémaux sur des surfaces compactes sans bord

Soit Σ une surface compacte sans bord connexe. Notons γ son genre, sachant que le genre d'une surface non orientable est le genre de son revêtement double orientable de sorte que la caractéristique d'Euler χ et le genre soient reliés par l'égalité $\chi = 1 - \gamma$. On s'intéresse aux invariants topologiques suivants :

$$\Lambda_k^g(\gamma) = \sup_g \lambda_k(g) A_g(\Sigma)$$

pour les surfaces orientables de genre γ ,

$$\Lambda_k^{no}(\gamma) = \sup_g \lambda_k(g) A_g(\Sigma)$$

pour les surfaces non orientables de genre γ , où $A_g(\Sigma)$ est l'aire de la surface. Yang et Yau [113] ont donné une borne sur ces invariants pour $k = 1$, qui ne dépendent que de γ en montrant en particulier qu'ils sont toujours finis :

$$\Lambda_1^o(\gamma) \leq 8\pi \left\lceil \frac{\gamma + 3}{2} \right\rceil \quad (5)$$

où l'inégalité n'est pas optimale sauf dans le cas $\gamma = 0$: c'est le théorème 5 de Hersch. Plus tard, Li et Yau [74] ont simplifié la preuve grâce à la notion de volume conforme qu'on définira à la section 1.4. Le volume conforme leur permet aussi de donner une borne pour les surfaces non orientables et Karpukhin [63] a amélioré cette borne :

$$\Lambda_1^{no}(\gamma) \leq 16\pi \left\lceil \frac{\gamma + 3}{2} \right\rceil .$$

Korevaar a démontré une généralisation de ces résultats pour $k > 1$:

Théorème 7 (Korevaar [67]). *Il existe une constante universelle $C > 0$ telle que*

$$\Lambda_k^o(\gamma) \leq Ck(\gamma + 1) \text{ et } \Lambda_k^{no}(\gamma) \leq Ck(\gamma + 1) \quad (6)$$

Dès lors, deux questions naturelles apparaissent sur ces invariants :

- 1 Peut-on calculer la valeur exacte de $\Lambda_k^o(\gamma)$ et de $\Lambda_k^{no}(\gamma)$? Si non, peut-on en donner des encadrements ?
- 2 La borne supérieure dans $\Lambda_k^o(\gamma)$ et $\Lambda_k^{no}(\gamma)$ est-elle atteinte ? Si oui, pour quelles métriques ?

Nous pouvons rassembler les résultats existants ou conjecturés dans le tableau suivant :

Valeur propre	Valeur Métriques maximales	Références et Commentaires
$\Lambda_1^o(0)$	8π S^2 , métrique ronde	[54]
$\Lambda_1^o(1)$	$\frac{8\pi^2}{\sqrt{3}} \simeq 14,510\pi$ $\mathbb{T}^2(\frac{1}{2}, \frac{\sqrt{3}}{2})$, tore équilatéral plat	[83]
$\Lambda_1^o(2)$	16π Famille de métriques maximales	[56] Résultats conjecturés
$\Lambda_2^o(0)$	16π Pas de métriques maximales	[84, 92]
$\Lambda_1^{no}(0)$	12π $\mathbb{R}P^2$, métrique ronde	[74]
$\Lambda_1^{no}(1)$	$12\pi E(\frac{2\sqrt{2}}{3}) \simeq 13,365\pi$ \mathbb{K}, g_0	[57, 32]

où sur la bouteille de Klein \mathbb{K} , la métrique maximale est une métrique de révolution qui vaut

$$g_0 = \frac{9 + (1 + 8 \cos^2 v)^2}{1 + 8 \cos^2 v} \left(du^2 + \frac{dv^2}{1 + 8 \cos^2 v} \right)$$

pour $0 \leq u \leq \frac{\pi}{2}$ et $0 \leq v < \pi$, et $E(\frac{2\sqrt{2}}{3})$ est l'intégrale elliptique complète de deuxième espèce évaluée en $\frac{2\sqrt{2}}{3}$.

Noter que dans le cas de la sphère, du tore, du plan projectif et de la bouteille de Klein, il existe une métrique maximale pour la première valeur propre et elle est unique à isométrie près. Pour la deuxième valeur propre sur la sphère, il n'existe pas de métrique maximale. Dans tous les cas non répertoriés dans le tableau, nous n'avons aucune réponse ni conjecture exactes sur les questions 1 et 2.

Cependant, pour la question 1, en plus du majorant (5), nous pouvons donner un minorant pour γ suffisamment grand. En fait, ce minorant vient en se focalisant sur les métriques hyperboliques : notons $\Lambda_1^h(\gamma)$ la borne supérieure parmi toutes les métriques hyperboliques des surfaces orientables de genre γ de la première valeur propre non nulle du Laplacien. Nous savons par un résultat de Buser [13] que

$$\limsup_{\gamma \rightarrow +\infty} \Lambda_1^h(\gamma) \leq \frac{1}{4}.$$

En combinant un résultat de Buser, Burger et Dodziuk [14] et celui de Brooks et Makover [9], théorème 1.2, nous obtenons

$$\liminf_{\gamma \rightarrow +\infty} \Lambda_1^h(\gamma) \geq C, \quad (7)$$

où C est la constante de Selberg. Par un théorème de Selberg [105], nous savons que $C \geq \frac{3}{16}$. Cette borne a été améliorée par Luo, Rudnick et Sarnak [75] $C \geq \frac{171}{784}$ pour approcher la fameuse conjecture de Selberg $C = \frac{1}{4}$. Nous obtenons ainsi

$$4\pi C \leq \liminf_{\gamma \rightarrow +\infty} \frac{\Lambda_1^o(\gamma)}{\gamma} \leq \limsup_{\gamma \rightarrow +\infty} \frac{\Lambda_1^o(\gamma)}{\gamma} \leq 4\pi.$$

La première inégalité vient de (7) et du théorème de Gauss-Bonnet et la dernière inégalité vient de la borne de Yang et Yau (5). Nous obtenons alors comme nous l'avons dit un minorant pour γ suffisamment grand :

$$\Lambda_1^o(\gamma) \geq \frac{3\pi}{4}(\gamma - 1). \quad (8)$$

Nous pouvons enfin comparer les $\Lambda_k^o(\gamma)$ comme fonctions de γ et k grâce à l'inégalité :

$$\lambda_k^o(\gamma) \geq \Lambda_{i_1}^o(\gamma_1) + \dots + \Lambda_{i_s}^o(\gamma_s) \quad (9)$$

pour tous $\gamma_1 + \dots + \gamma_s \leq \gamma$ et $i_1 + \dots + i_s = k$. Elle est démontrée par des méthodes de recollement par Colbois et El Soufi, [22]. Une inégalité similaire pour les surfaces non orientables y est donnée. Pour démontrer (9), il suffit de savoir démontrer les deux inégalités suivantes :

$$\Lambda_k^o(\gamma) \geq \Lambda_k^o(\gamma - 1) \quad (10)$$

$$\Lambda_k^o(\gamma) \geq \Lambda_{i_1}^o(\gamma_1) + \Lambda_{i_2}^o(\gamma_2) \quad (11)$$

Pour (10), on prend une métrique qui est presque maximale pour la caractérisation variationnelle de $\Lambda_k^o(\gamma - 1)$ sur une surface de genre $\gamma - 1$. On ajoute à la surface une anse d'aire très petite. On teste la métrique ainsi construite sur une nouvelle surface de genre γ pour le problème variationnel de $\Lambda_k^o(\gamma)$. On obtient l'inégalité (10) à une constante près aussi petite qu'on veut. Pour (11), on prend une métrique qui est presque maximale pour la caractérisation variationnelle de $\Lambda_{i_1}^o(\gamma_1)$ sur une surface de genre γ_1 et une métrique presque maximale pour la caractérisation variationnelle de $\Lambda_{i_2}^o(\gamma_2)$ sur une surface de genre γ_2 . On fait une somme connexe des deux surfaces grâce à un cylindre d'aire très petite. On teste la métrique ainsi construite sur une nouvelle surface de genre γ pour le problème variationnel de $\Lambda_k^o(\gamma)$. On obtient l'inégalité (11) à une constante près aussi petite qu'on veut.

En particulier, nous obtenons que

$$\Lambda_k^o(0) \geq 8\pi k = k\Lambda_1^o(0)$$

ce qui est une égalité lorsque $k = 1$ (Théorème 5) et $k = 2$ (Théorème 6). Il est conjecturé que cette inégalité est une égalité pour tous k où le cas d'égalité est dégénéré : c'est l'union disjointe de k sphères de même aire donnée dans la preuve de (9). Sur les tores nous obtenons l'inégalité

$$\Lambda_k^o(1) \geq \frac{8\pi^2}{\sqrt{3}} + 8\pi(k-1) = \Lambda_1^o(1) + (k-1)\Lambda_1^o(0).$$

Nous pouvons aussi nous demander si cette inégalité est une égalité. Le cas dégénéré est ici l'union d'un tore avec $k-1$ sphères. Enfin, notons qu'en genre 2, avec la conjecture $\Lambda_1^o(2) = 16\pi$ de [56], nous obtiendrions

$$\Lambda_2^o(2) \geq 2\Lambda_1^o(1) = \frac{16\pi^2}{\sqrt{3}} > 24\pi = \Lambda_1^o(2) + \Lambda_1^o(0)$$

et si l'inégalité est une égalité, le cas dégénéré ne correspond plus à l'union de la surface avec des sphères mais à une union de deux tores équilatéraux plats.

Des réponses à la question 2 seront données dans la section 2.

1.4 Spectre conforme

Soit (M, g) une variété Riemannienne compacte connexe sans bord de dimension m . Pour $k \in \mathbb{N}$, définissons la k -ième valeur propre conforme comme

$$\Lambda_k(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_k(M, \tilde{g}) V_{\tilde{g}}(M)^{\frac{2}{m}}$$

où $[g]$ désigne la classe conforme de g , c'est à dire l'ensemble des métriques qui sont multiples de g par une fonction strictement positive et de classe \mathcal{C}^∞ sur M . Remarquer ici que seule la borne supérieure peut avoir un intérêt car comme nous l'avons vu dans la preuve de la Proposition 1, la borne inférieure est nulle. Par un résultat de Korevaar [67], nous savons que cet invariant conforme est toujours fini. Il donne une majoration qui ne dépend que de k et de la classe conforme de g , raffiné plus tard par Hassannezhad [49] comme :

$$\Lambda_k(M, [g]) \leq A_m V([g])^{\frac{2}{m}} + B_m k^{\frac{2}{m}} \quad (12)$$

où en notant Ric_{g_0} la courbure de Ricci d'une métrique g_0 ,

$$V([g]) = \inf\{V_{g_0}(M); g_0 \in [g], Ric_{g_0} \geq -(m-1)\}$$

est un invariant ne dépendant que de la classe conforme de g et A_m et B_m sont des constantes qui ne dépendent que de la dimension de la variété M .

Sachant que $\Lambda_k(M, [g])$ est fini, deux questions naturelles apparaissent pour cet invariant :

- 1 Peut-on calculer la valeur exacte de $\Lambda_k(M, [g])$? Si non, peut-on en donner des encadrements ?
- 2 La borne supérieure dans $\Lambda_k(M, [g])$ est-elle atteinte ? Si oui, pour quelles métriques ?

Nous nous intéressons aux valeurs propres conformes car elles présentent un intérêt en elles-même pour plusieurs raisons

- En dimension $m \geq 3$, maximiser sur l'ensemble des métriques n'est pas intéressant car on peut toujours trouver une métrique g sur M telle que $\lambda_k(M, g)V_g(M)$ est aussi grand qu'on veut [20]. Une restriction pertinente est donc une classe conforme fixée d'après le théorème de Korevaar [67].
- En dimension 2, le passage par cet invariant conforme est un outil important pour démontrer des résultats sur $\Lambda_k(\gamma)$. En effet, maximiser parmi des métriques conformes entre elles est un problème variationnel sur un espace de fonctions, beaucoup plus simple qu'un espace de métriques. De plus, comme le Laplacien est un invariant conforme en dimension 2, l'énergie de Dirichlet est un invariant conforme, ce qui facilite la gestion du quotient de Rayleigh. D'ailleurs, le Théorème 7 de Korevaar est une conséquence de la finitude de $\Lambda_k(\Sigma, [g])$ pour toute surface riemannienne (Σ, g) . L'inégalité (12) se traduit sur les surfaces car par le théorème de Gauss-Bonnet, $V([g]) \leq 4\pi(\gamma - 1)$ pour les surfaces de genre $\gamma \geq 2$ et $V([g]) = 0$ pour les surfaces de genre 0 et 1. Cela donne

$$\Lambda_k(\gamma) \leq \alpha\gamma + \beta k$$

pour des constantes universelles α et β . Comme nous le verrons en Section 2.2, la restriction à une classe conforme donnée aura aussi un intérêt pour répondre à la question 2 énoncée juste après le Théorème 7.

- L'invariant $\Lambda_k(M, [g])$ est relié à un autre invariant pour $k = 1$. En effet, il existe une autre majoration que (12) précédemment trouvée par Li et Yau par le volume conforme qu'ils introduisent dans [74] en dimension 2 et par El Soufi et Ilias en dimension $m \geq 3$ [33]. Elle s'écrit :

$$\Lambda_1(M, [g]) \leq mV_c(n, M, [g])^{\frac{2}{m}}$$

où V_c est le volume conforme :

$$V_c(n, M, [g]) = \inf_{\substack{\psi: (M, g) \rightarrow (S^n, g_s) \\ \text{conforme}}} \sup_{\theta \in \text{Conf}(S^n)} V_{g_s}(\theta \circ \psi(M))$$

avec g_s la métrique standard de la sphère ronde et $\text{Conf}(S^n)$ l'ensemble des difféomorphismes conformes de la sphère ronde. Par convention, $V_c = +\infty$ s'il n'existe pas d'immersion conforme de (M, g) dans (S^n, g_s) . Le cas d'égalité est donné par une immersion minimale par les premières fonctions propres dans une sphère S^n , qui seront vues dans la section 2.1. Ainsi, majorer $\lambda_1(M, \tilde{g})V_{\tilde{g}}(M)^{\frac{2}{m}}$ dans une classe conforme fixée peut donner lieu à l'existence d'objets géométriques particuliers.

Des valeurs exactes sont données pour la question 1 concernant les petites valeurs propres conformes dans des classes conformes simples :

- $\Lambda_1(\mathbb{S}^m, [g_s]) = m\sigma_m^{\frac{2}{m}}$ où σ_m est le volume de la sphère euclidienne de rayon 1 et de dimension m .
- $\Lambda_1(\mathbb{RP}^m, [g_s]) = 2^{\frac{m-2}{m}}(m+1)\sigma_m^{\frac{2}{m}}$
- $\Lambda_1(\mathbb{CP}^d, [g_s]) = 4\pi(d+1)(d!)^{-\frac{1}{d}}$
- $\Lambda_1(\mathbb{HP}^d, [g_s]) = 8\pi(d+1)((2d+1)!)^{-\frac{1}{2d}}$

où g_s désigne les métriques standard sur les variétés précédemment citées. Sur la sphère, $\Lambda_1(\mathbb{S}^m, [g_s]) = m\sigma_m^{\frac{2}{m}}$ est atteint si et seulement si la métrique est ronde. C'est une généralisation immédiate du théorème de Hersch (Théorème 5). Sur les tores de dimension 2, il est démontré dans [36] que certains tores plats sont maxima dans leur classe conforme.

Concernant la deuxième valeur propre sur la sphère, nous avons obtenu au cours de la thèse [92] une généralisation du Théorème 6 en dimension supérieure. La démonstration est donnée dans le chapitre 1.

Théorème 8 (P. [92]). *Soit $m \geq 2$ et $g \in [g_s]$ une métrique conforme à la métrique ronde. Alors*

$$\lambda_2(\mathbb{S}^m, g)V_g(\mathbb{S}^m) < K_m m(2\sigma_m)^{\frac{2}{m}}$$

où K_m est une constante indépendante de $g \in [g_s]$ donnée par

$$K_m = \frac{m+1}{m} \left(\frac{\Gamma(m) \Gamma(\frac{m+1}{2})}{\Gamma(m+\frac{1}{2}) \Gamma(\frac{m}{2})} \right)^{\frac{2}{m}}$$

et satisfait $K_2 = 1$, $1 < K_m < 1,04$ pour $m \geq 3$ et $\lim_{m \rightarrow +\infty} K_m = 1$.

En dimension 2, c'est simplement le théorème 6. En dimension $m \geq 3$, ce résultat avait déjà été démontré dans [46] en dimensions impaires. Le théorème 8 unifie donc ici les résultats précédents.

Nous pouvons estimer le spectre conforme en le comparant à celui de la sphère standard

$$\Lambda_k(M, [g])^{\frac{m}{2}} \geq \Lambda_{k-j}(M, [g])^{\frac{m}{2}} + \sum_{p=1}^s \Lambda_{i_p}(\mathbb{S}^m, [g_s])^{\frac{m}{2}} \quad (13)$$

où $0 \leq j \leq k$ et $i_1 + \dots + i_s = j$. Pour $s = 1$ et $j = k$ et pour $s = k$ et $j = k$, le résultat devient :

$$\Lambda_k(M, [g]) \geq \Lambda_k(\mathbb{S}^m, [g_s]) \geq m\sigma_m^{\frac{2}{m}} k^{\frac{2}{m}} \quad (14)$$

Pour $s = 1$ et $j = 1$ le résultat devient :

$$\Lambda_k(M, [g])^{\frac{m}{2}} - \Lambda_{k-1}(M, [g])^{\frac{m}{2}} \geq \Lambda_1(\mathbb{S}^m, [g_s])^{\frac{m}{2}}. \quad (15)$$

Les inégalités (14) et (15) sont démontrées par Colbois et El Soufi dans [21] par des méthodes de recollement basées sur l'idée suivante : on peut adjoindre une sphère munie d'une métrique conforme à la métrique ronde à une variété riemannienne sans changer la classe conforme.

Pour la première inégalité dans (14), il suffit de prendre une métrique presque maximale pour $\Lambda_k(\mathbb{S}^m, [g_s])$, de l'adjoindre à une métrique dans la classe conforme de $[g]$ en lui faisant porter presque tout le volume et de tester la nouvelle métrique dans la caractérisation

variationnelle de $\lambda_k(M, [g])$. Ceci donne l'inégalité voulue à des paramètres près aussi petits qu'on veut. Pour l'inégalité dans (15), il suffit de prendre une métrique presque maximale pour $\Lambda_{k-1}(M, [g])$ et de lui adjoindre une sphère ronde, métrique maximale pour $\Lambda_1(\mathbb{S}^m, [g_s])$ en respectant les proportions de volume. Tester cette nouvelle métrique dans la caractérisation variationnelle de $\Lambda_k(M, [g])$ donne l'inégalité voulue à des paramètres près aussi petits qu'on veut. Noter que la preuve de (13) suit la même procédure. La deuxième inégalité dans (14) s'obtient en appliquant k fois l'inégalité de (15) sur la sphère.

En particulier, (14) donne l'inégalité

$$\Lambda_2(\mathbb{S}^m, [g_s]) \geq m (2\sigma_m)^{\frac{2}{m}}, \quad (16)$$

ce qui donne en dimension 2 grâce au Théorème 6, $\Lambda_2(\mathbb{S}^2, [g_s]) = 16\pi$.

Une question naturelle vient pour $\Lambda_2(\mathbb{S}^m, [g_s])$. Pouvons-nous améliorer l'encadrement donné par le Théorème 8 et l'inégalité (16) en réduisant le facteur K_m ? De manière surprenante, contre l'esprit des Théorèmes 2, 4 et 6 dans lesquels la deuxième valeur propre maximale est optimisée dans le cas dégénéré de deux formes disjointes de même volume associées à la première valeur propre maximale, nous n'avons pas égalité dans l'inégalité (16) pour $m \geq 3$. C'est un résultat de Druet :

Théorème 9 (Druet [28]). *Pour $m \geq 3$,*

$$\Lambda_2(\mathbb{S}^m, [g_s]) > m (2\sigma_m)^{\frac{2}{m}}.$$

Nous obtenons ainsi l'encadrement pour tout $m \geq 3$

$$m (2\sigma_m)^{\frac{2}{m}} < \Lambda_2(\mathbb{S}^m, [g_s]) \leq K_m m (2\sigma_m)^{\frac{2}{m}},$$

avec $K_2 = 1$, $1 < K_m < 1,04$ pour $m \geq 3$ et $\lim_{m \rightarrow +\infty} K_m = 1$.

L'inégalité (14) donne pour $k = 1$

$$\Lambda_1(M, [g]) \geq \Lambda_1(\mathbb{S}^m, [g_s]) = m\sigma_m^{\frac{2}{m}}.$$

Nous avons démontré au cours de la thèse un résultat de rigidité stipulant que cette inégalité est stricte sauf si $(M, [g])$ est conforme à la sphère standard $(\mathbb{S}^m, [g_s])$. La démonstration est donnée dans le chapitre 2 en dimension supérieure à 3 et dans le chapitre 3 en dimension 2.

Théorème 10 (P. [94]). *On a*

$$\Lambda_1(M, [g]) \geq \Lambda_1(\mathbb{S}^m, [g_s]) = m\sigma_m^{\frac{2}{m}},$$

avec égalité si et seulement si $(M, [g])$ est conforme à $(\mathbb{S}^m, [g_s])$.

En particulier, pour $m = 2$, nous avons

$$\Lambda_1(\Sigma, [g]) > 8\pi \quad (17)$$

pour toute surface Σ non difféomorphe à une sphère. Ce résultat permet de montrer l'existence de métriques maximales pour $\Lambda_1(\Sigma, [g])$ pour toute surface munie d'une classe conforme $(\Sigma, [g])$, comme nous le verrons dans la section suivante.

Une autre conséquence de ce résultat de rigidité est donnée pour le volume conforme :

Théorème 11 (P. [94]). *On a*

$$V_c(n, M, [g]) \geq V_c(S^m, [g_s]) = \sigma_m ,$$

avec égalité si et seulement si $(M, [g])$ est conforme à $(S^m, [g_s])$.

Ceci répond à une question de Li et Yau ouverte depuis 30 ans. Dans leur article original sur le volume conforme [74], ils posaient en effet deux questions :

- 1 Existe-t-il une application conforme $\psi : (M, g) \rightarrow (S, g_s)$ pour laquelle la borne inférieure dans la définition du volume conforme est atteinte ?
- 2 Si $V_c(n, M, [g]) = V_c(S^m, [g_s])$, (M, g) est-elle nécessairement conforme à $(S^m, [g_s])$?

Le théorème 11 donne donc une réponse à la question 2. En particulier, sur les surfaces, nous obtenons

$$V_c(n, \Sigma, [g]) > 4\pi \tag{18}$$

si $(\Sigma, [g])$ n'est pas difféomorphe à une sphère. Ce résultat ouvre aussi une perspective concernant la question 1. En effet, en général, une inégalité stricte comme (18) permet d'éliminer des phénomènes de concentration des suites minimisantes. C'est exactement le rôle que joue l'inégalité stricte (17) pour l'existence d'une métrique maximale pour $\lambda_1(\tilde{g})A_{\tilde{g}}(\Sigma)$ lorsque la métrique \tilde{g} est conforme à une métrique g de référence, comme on le verra dans le théorème 14. Voir le papier de Rivière [99] pour les avancées sur cette question 1 pour le volume conforme.

2 Surfaces extrémales pour les valeurs propres de Laplace

2.1 Lien avec les surfaces minimales à valeurs dans une sphère

Dans cette section, Σ désigne une surface compacte sans bord. Les valeurs propres du Laplacien sont liées à la théorie des surfaces minimales. On dit que la surface $\Phi(\Sigma)$ associée à une immersion $\Phi : (\Sigma, g) \rightarrow S^n$ est minimale (ou que l'immersion Φ est minimale) si la courbure moyenne de l'immersion est nulle. Un vieux résultat de Takahashi donne la correspondance suivante :

Théorème 12 (Takahashi [108]). *Soit $\Phi : (\Sigma, g) \rightarrow S^n$ une immersion isométrique. Alors Φ est minimale si et seulement si toutes ses coordonnées sont des fonctions propres associées à une valeur propre donnée pour la métrique g .*

En fait, le résultat est plus précis si on ne suppose pas au préalable que l'immersion est isométrique : une immersion est isométrique et minimale à valeurs dans une sphère si et seulement si la métrique induite est extrémale pour une valeur propre donnée sur la surface. Ici, il faut donner un sens à extrémal. Cela signifie qu'elle vérifie une équation d'Euler-Lagrange pour un problème variationnel donné. Dans le cadre classique, c'est simplement exprimer le fait que la dérivée est nulle en un point critique. Ici, la fonctionnelle $g \mapsto A_g(\Sigma)\lambda_k(\Sigma, g)$ n'est pas \mathcal{C}^1 en les métriques où la valeur propre λ_k est multiple. Ainsi, nous n'avons pas de caractérisation variationnelle dans un cadre classique. Dans notre cadre, on dit que la métrique est extrémale si 0 appartient au sous-différentiel de la fonctionnelle dans le sens précisé par la définition suivante :

Définition. On appelle dérivée directionnelle de $\lambda : g \mapsto A_g(\Sigma)\lambda_k(\Sigma, g)$

$$\lambda'(g, h) = \frac{d}{dt}\bigg|_{t=0^+} A_{g+th}(\Sigma)\lambda_k(\Sigma, g+th)$$

où g est une métrique sur M et h une 2-forme symétrique sur M . On appelle sous-différentiel de $\lambda : g \mapsto A_g(\Sigma)\lambda_k(\Sigma, g)$ en g l'ensemble suivant :

$$\partial\lambda(g) = \text{Conv}\{h \in \mathcal{S}(M); \forall \tilde{h} \in \mathcal{S}(M), \langle h, \tilde{h} \rangle \leq \lambda'(g, \tilde{h})\}$$

où $\mathcal{S}(M)$ est l'ensemble des 2-formes symétriques sur M et pour $h, \tilde{h} \in \mathcal{S}(M)$,

$$\langle h, \tilde{h} \rangle = \int_{\Sigma} (h, \tilde{h})_g dv_g$$

où $(\cdot, \cdot)_g$ désigne le produit scalaire entre 2-formes via la métrique g de sorte que $(g, \cdot)_g$ est la trace sur g d'une 2-forme.

Dans notre cas, cette définition correspond aux dérivées directionnelles généralisées et au sous-différentiel généralisé au sens de Clarke ([18], chapitre 10) d'une application localement lipschitzienne définie sur un espace de Banach. On a besoin de cette définition parce que même si la fonction $t \mapsto A_{g+th}(\Sigma)\lambda_k(\Sigma, g+th)$ n'est pas \mathcal{C}^1 en 0, elle admet une dérivée à gauche et une dérivée à droite avec

$$\begin{aligned} \lambda'(g, h) &= \inf_{\phi \in E_k(g)} \frac{\int_{\Sigma} \left(\lambda_k(\Sigma, g)(1 - \phi^2)\frac{g}{2} + \frac{|\nabla\phi|_g^2 g}{2} - d\phi \otimes d\phi, h \right)_g dv_g}{\int_{\Sigma} \phi^2 dv_g}, \\ \lambda'(g, -h) &= \sup_{\phi \in E_k(g)} \frac{\int_{\Sigma} \left(\lambda_k(\Sigma, g)(1 - \phi^2)\frac{g}{2} + \frac{|\nabla\phi|_g^2 g}{2} - d\phi \otimes d\phi, h \right)_g dv_g}{\int_{\Sigma} \phi^2 dv_g}. \end{aligned}$$

La preuve de ces égalités se trouve par exemple dans Fraser-Schoen [39]. Le sous différentiel vaut alors

$$\text{Conv} \left\{ \lambda_k(\Sigma, g)(1 - \phi^2)\frac{g}{2} + \frac{|\nabla\phi|_g^2 g}{2} - d\phi \otimes d\phi; \phi \in E_k(g) \right\}$$

où $E_k(g)$ désigne l'espace propre associé à la k -ième valeur propre du Laplacien. Pour trouver ces résultats, nous pouvons aussi utiliser de la formule de Dunskin ([18], 10.22). Définissons alors les métriques extrémales :

Définition. On dit que g est extrémale pour la k -ième valeur propre si elle vérifie l'une des assertions équivalentes suivantes :

(i) Pour toute 2-forme symétrique h ,

$$\left(\frac{d}{dt}\bigg|_{t=0^+} A_{g+th}(\Sigma)\lambda_k(\Sigma, g+th) \right) \left(\frac{d}{dt}\bigg|_{t=0^-} A_{g+th}(\Sigma)\lambda_k(\Sigma, g+th) \right) \leq 0.$$

(ii) Le sous-différentiel de $\lambda : g \mapsto A_g(\Sigma)\lambda_k(\Sigma, g)$ contient 0 :

$$0 \in \text{Conv} \left\{ \lambda_k(\Sigma, g)(1 - \phi^2)\frac{g}{2} + \frac{|\nabla\phi|_g^2 g}{2} - d\phi \otimes d\phi; \phi \in E_k(g) \right\}.$$

La démonstration de cette équivalence entre (i) et (ii) est donnée par exemple dans [39] par un argument utilisant le théorème de Hahn-Banach. Le sous-différentiel apparaît dans (ii), qui est la formulation d'Euler-Lagrange pour le sous-différentiel de λ comme souligné dans [18]. Il donne l'existence de $\phi_1, \dots, \phi_m \in E_k(g)$, m fonctions propres indépendantes telles que

$$0 = \lambda(1 - \Phi^2) \frac{g}{2} + \sum_{i=1}^m d\phi_i \otimes d\phi_i - \frac{|\nabla \Phi|_g^2}{2} g \quad (19)$$

où $|\Phi|^2 = \sum_{i=1}^m \phi_i^2$ et $|\nabla \Phi|_g^2 = \sum_{i=1}^m |\nabla \phi_i|_g^2$. En traçant sur g , on obtient $|\Phi|^2 = 1$. En calculant $\Delta_g |\Phi|^2$ et en utilisant l'équation aux valeurs propres, on obtient que $|\nabla \Phi|_g^2 = \lambda$ et donc que Φ est harmonique à valeurs dans S^{m-1} , c'est à dire

$$\Delta_g \Phi = |\nabla \Phi|_g^2 \Phi.$$

Ainsi, avec (19),

$$\frac{\lambda}{2} g = \sum_{i=1}^m d\phi_i \otimes d\phi_i$$

ce qui montre que l'immersion est isométrique quitte à dilater g par $\frac{\lambda}{2}g$. On obtient qu'elle est minimale par le Théorème 12 de Takahashi.

De nombreux travaux d'existence d'immersions minimales dans des sphères S^n ont été effectués depuis Lawson pour $n = 3$ [72] et Bryant pour $n = 4$ [11]. Pourtant, la classification des immersions minimales dans des sphères est loin d'être aboutie, même pour les plongements. Par exemple, Yau a conjecturé [114] que les plongements minimaux dans S^3 vérifient tous que leurs coordonnées sont des premières fonctions propres. Récemment, Brendle [7] a montré la conjecture de Lawson qui stipule que le seul plongement minimal d'un 2-tore dans S^3 est le tore de Clifford (dont on sait depuis Montiel et Ros [80] qu'il est le seul tore minimal immergé dans S^3 par les premières fonctions propres). Chercher les points critiques de λ_k pour k fixé est un autre point de vue pour l'étude des immersions minimales dans des sphères.

On peut dans un premier temps rechercher le rang k des valeurs propres critiques associées aux immersions minimales existantes. On peut le faire pour les immersions minimales classiques.

- La sphère standard : le plongement de S^2 dans \mathbb{R}^3 d'aire 4π .
- Le plongement de Veronese : le plongement de $\mathbb{R}P^2 = S^2/\{id, \sigma\}$ dans \mathbb{R}^5 , quotient par l'antipodie $\sigma(x) = -x$ de l'application $\psi : S^2 \rightarrow \mathbb{R}^5$ définie par

$$\psi(x, y, z) = \sqrt{3} \left(xy, xz, yz, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2) \right)$$

d'aire 6π .

- Le tore de Clifford : le plongement de $T^2(0,1) = \mathbb{R}^2/\mathbb{Z}^2$, quotient de l'application $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^4 = \mathbb{C}^2$ définie par

$$\psi(x, y) = \frac{1}{\sqrt{2}} (e^{2i\pi x}, e^{2i\pi y})$$

d'aire $2\pi^2$.

- Le tore équilatéral plat : le plongement de $\mathbb{T}^2(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \mathbb{R}^2 / \Lambda_e$ où Λ_e est le réseau engendré par $(0, 1)$ et $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, quotient de l'application $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^6 = \mathbb{C}^3$ définie par

$$\psi(x, y) = \frac{1}{\sqrt{3}} \left(e^{\frac{4i\pi y}{\sqrt{3}}}, e^{2i\pi(x - \frac{y}{\sqrt{3}})}, e^{2i\pi(x + \frac{y}{\sqrt{3}})} \right)$$

d'aire $\frac{4\pi^2}{\sqrt{3}}$.

Pour tous ces plongements minimaux, les coordonnées sont des premières fonctions propres associées à la métrique induite.

D'autres exemples ont été traités par Penskoï [89], [90] et Karpukhin [59], [61], [60] donnant notamment les rangs des valeurs propres critiques associées à des tores de Lawson et des tores d'Otsuki.

2.2 Existence et régularité de métriques maximales

Comme tout maximiseur est point critique et comme les fonctions $\Lambda_k^o(\gamma)$ sont finies par le théorème 7, il est naturel de chercher les métriques extrémales qui sont les maximiseurs pour $\Lambda_k^o(\gamma)$ s'ils existent. Avant de donner le principal résultat de la thèse, rappelons les bornes données sur $\Lambda_k^o(\gamma)$ par des méthodes de recollement (9) :

$$\Lambda_k^o(\gamma) \geq \max_{\substack{i_1 + \dots + i_s = k \\ \forall m, i_m \geq 1 \\ \gamma_1 + \dots + \gamma_s \leq \gamma \\ \gamma_1 < \gamma \text{ si } s=1}} \Lambda_{i_1}^o(\gamma_1) + \dots + \Lambda_{i_s}^o(\gamma_s). \quad (20)$$

Nous obtenons un théorème d'existence, prouvé dans le chapitre 4 :

Théorème 13 (P.). *Soit Σ une surface orientable compacte sans bord de genre γ . Si l'inégalité (20) est stricte, alors il existe une métrique g sur Σ qui est C^∞ sauf peut-être en un nombre fini de points de singularité conique telle que $\Lambda_k(\gamma) = \lambda_k(g) \text{Vol}_g(\Sigma)$. De plus, cette métrique est le tiré en arrière d'une immersion minimale de Σ dans une sphère S^n par des k -èmes fonctions propres. Enfin, sous cette condition, l'ensemble des métriques maximales est compact.*

Noter que l'hypothèse disant que (20) est stricte est nécessaire car par exemple, on sait que $\Lambda_2^o(0) = 2\Lambda_1^o(0) = 16\pi$ et que le maximum n'est pas atteint d'après le théorème (6). Noter aussi le fait que la métrique maximale n'est pas forcément C^∞ partout : l'immersion minimale peut avoir des points de branchement. C'est d'ailleurs ce qui est conjecturé pour $k = 1$ et $\gamma = 2$ [56] : les métriques maximales ont des singularités coniques.

Dans le cas $k = 1$, l'hypothèse d'inégalité stricte pour (20) faite dans le théorème 13 se traduit par $\Lambda_1^o(\gamma) > \Lambda_1^o(\gamma - 1)$. Cette condition est vraie pour $\gamma = 1$, [83], et conjecturée pour $\gamma = 2$, [56]. En fait, elle est vraie une infinité de fois d'après la borne inférieure (8). On obtient le

Corollaire (P.). *Soit Σ une surface orientable compacte sans bord de genre γ . Il existe une immersion minimale de Σ dans une sphère par des premières fonctions propres pour une infinité de genres γ .*

Ce corollaire est un premier pas vers la question de la classification des immersions minimales par des premières fonctions propres de Montiel et Ros [80]. Ce résultat d'existence est

d'autant plus spectaculaire que les immersions minimales par des premières fonctions propres sont rares : il n'y en a qu'au plus une dans une classe conforme donnée d'après [80].

La preuve du théorème 13 se décompose en deux parties et repose sur un autre théorème en remarquant que maximiser la k -ième valeur propre parmi toutes les métriques d'aire 1 revient à la maximiser d'abord parmi toutes les métriques d'aire 1 dans une classe conforme donnée puis parmi toutes les classes conformes. C'est pour cela qu'on introduit l'invariant conforme

$$\Lambda_k(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \lambda_k(\Sigma, \tilde{g}) \text{Vol}_{\tilde{g}}(\Sigma)$$

sur toute surface Σ munie d'une métrique g . $[g]$ désigne la classe conforme de g . On en déduit

$$\Lambda_k^0(\gamma) = \sup_{[g]} \Lambda_k(\Sigma, [g]) .$$

Enonçons le théorème de maximisation de $\Lambda_k(\Sigma, [g])$ utilisé pour démontrer le théorème 13. Il est prouvé dans le chapitre 4.

Théorème 14 (P.). *Soit (Σ, g) une surface Riemannienne compacte sans bord et $k \geq 1$. Si (13) est stricte, alors il existe une métrique maximale $\tilde{g} \in [g]$ qui est \mathcal{C}^∞ sauf peut-être en un nombre fini de points de singularité conique telle que $\Lambda_k(\Sigma, [g]) = \lambda_k(\Sigma, \tilde{g}) \text{Vol}_{\tilde{g}}(\Sigma)$. De plus, il existe une famille de fonctions propres orthogonales associées à $\lambda_k(\Sigma, \tilde{g})$ formant une application harmonique à valeurs dans une sphère S^n . Enfin, sous cette condition, l'ensemble des métriques maximales est compact.*

Dans le cas $k = 1$, l'hypothèse se lit $\Lambda_1(\Sigma, [g]) > \Lambda_1(S^2, [g_s]) = 8\pi$, ce qui est vrai dès que Σ n'est pas difféomorphe à une sphère d'après le théorème 10. L'hypothèse est aussi sans doute vraie dans le cas $k = 2$, pour certaines surfaces de genre 2. En effet, grâce à (20) et aux valeurs $\Lambda_1(1) = \frac{8\pi^2}{\sqrt{3}}$ et $\Lambda_1(2) = 16\pi$ respectivement obtenues dans [83] et conjecturée dans [56] on obtient que $\Lambda_2(2) \geq 2\Lambda_1(1) = \frac{16\pi^2}{\sqrt{3}} > 24\pi = \Lambda_1(2) + \Lambda_1(0)$. Cela montre en particulier qu'il existerait un ouvert de classes conformes qui satisfont l'inégalité (13) stricte sur des surfaces de genre 2 pour $k = 2$, c'est à dire $\Lambda_2(\Sigma, [g]) > \Lambda_1(\Sigma, [g]) + 8\pi$. Ainsi, le théorème 14 s'applique dans ces cas et on obtient des métriques maximales \mathcal{C}^∞ sauf peut-être en un nombre fini de points de singularité conique.

Si le théorème 14 s'applique, il existe une métrique maximale \tilde{g} pour $\Lambda_k(\Sigma, [g])$ et le facteur conforme associé à \tilde{g} est $|\nabla \Phi|_{\tilde{g}}^2$ où $\Phi : \Sigma \rightarrow S^n$ est une application harmonique dont les coordonnées sont des fonctions propres associées à $\lambda_k(\tilde{g})$. Les singularités coniques apparaissent naturellement comme zéros de $|\nabla \Phi|_{\tilde{g}}^2$. Ils sont isolés comme c'est démontré par Salamon [101].

Signalons un dernier résultat qui prouve une conjecture de Friedlander et Nadirashvili [42] donnant l'infimum de $\Lambda_1(\sigma, [g])$ parmi toutes les classes conformes sur une surface orientable Σ . Il est démontré dans le chapitre 3.

Théorème 15 (P. [91]). *Soit Σ une surface compacte orientable. On a :*

$$\inf_{[g]} \Lambda_1(\Sigma, [g]) = 8\pi$$

et l'infimum n'est jamais atteint sauf sur la sphère.

Ce théorème montre que l'infimum parmi toutes les classes conformes n'est pas un invariant intéressant sur les surfaces orientables. En particulier, il ne donne pas de nouvelles métriques extrémales.

2.3 Elements de démonstration du théorème 14

La démonstration est donnée dans le chapitre 4 de la thèse et dans le chapitre 3 dans le cas plus simple de la première valeur propre. Nous en donnons ici un résumé pour faciliter la lecture. Soit (Σ, g) une surface Riemannienne. Rappelons d'abord que de même que dans la section 2.1 si une métrique $\tilde{g} = e^{2u}g$ avec u une fonction de classe C^∞ est maximale pour $\Lambda_k(\Sigma, [g])$, elle vérifie les deux propriétés équivalentes suivantes :

(i) Pour toute fonction v ,

$$\left(\frac{d}{dt} \Big|_{t=0^+} A_{g_t}(\Sigma) \lambda_k(\Sigma, g_t) \right) \left(\frac{d}{dt} \Big|_{t=0^-} A_{g_t}(\Sigma) \lambda_k(\Sigma, g_t) \right) \leq 0 ,$$

où $g_t = (1 + tv)\tilde{g}$.

(ii) Le sous-différentiel contient 0 :

$$0 \in \text{Conv} \{ \lambda_k(\Sigma, \tilde{g})(1 - \phi^2); \phi \in E_k(\tilde{g}) \} .$$

En traduisant (ii), il existe $\Phi = (\phi_1, \dots, \phi_m)$ une application à valeurs dans une sphère dont les coordonnées sont des fonctions propres associées à la métrique $\tilde{g} = e^{2u}g$, c'est à dire :

$$\begin{cases} \Delta_g \phi_i = \Lambda_k(\Sigma, [g]) e^{2u} \phi_i \\ |\Phi|^2 = 1 \end{cases} \quad (21)$$

On peut dire que c'est l'équation d'Euler-Lagrange associée à notre problème variationnel. En calculant $\Delta |\Phi|^2 = 0$, on obtient que Φ satisfait l'équation des applications harmoniques à valeurs dans une sphère et que le facteur conforme de \tilde{g} par rapport à g est la densité d'énergie de l'application harmonique :

$$\begin{cases} \Delta_g \Phi = |\nabla \Phi|_g^2 \Phi \\ \tilde{g} = \frac{|\nabla \Phi|_g^2}{\Lambda_k(\Sigma, [g])} g \end{cases} \quad (22)$$

Pour démontrer le théorème 14, l'approche classique serait de prendre une suite maximisante de facteurs conformes $\{e^{2u_\epsilon}\}$ de les faire converger dans un espace plus gros que C^∞ où on peut avoir de la compacité et de tenter d'obtenir de la régularité pour la limite grâce à une équation d'Euler-Lagrange. Ici l'espace naturel à choisir est l'ensemble des mesures de probabilité $\mathcal{M}_1(M)$ sur M munie de la topologie faible étoile car la suite $\{e^{2u_\epsilon}\}$ est de norme 1 dans $L^1(M)$. Dans ce cas, nous ne pouvons pas obtenir d'équation d'Euler-Lagrange car il n'existe pas a priori de fonctions propres associées à une mesure quelconque. En fait, l'objet limite a peu de chances d'être régulier car λ_k est une fonctionnelle très peu régularisante. Même si l'objet limite était régulier, on peut imaginer ne pas avoir meilleure convergence qu'une convergence faible étoile au sens des mesures car λ_k ne voit pas les faibles perturbations d'une suite de facteurs conformes. Ce constat nous pousse à ne choisir qu'une suite maximisante particulière qui sera régularisée a priori grâce à l'utilisation de l'opérateur de la chaleur. Cette idée a été donnée par Fraser et Schoen, lorsqu'ils ont traité le problème analogue sur la première valeur propre de Steklov. La construction de cette suite maximisante fait l'objet de la première étape de la démonstration :

Etape 1 : Construction d'une suite maximisante.

Pour cette étape nous renvoyons à la section 4.3, page 93. Pour une mesure de Radon positive $\nu \in \mathcal{M}(M)$, notons $K_\epsilon[\nu]$ la solution en temps ϵ de l'équation de la chaleur associée au Laplacien Δ_g telle que

$$K_\epsilon[\nu]dv_g \rightarrow_\star \nu \text{ lorsque } \epsilon \rightarrow 0.$$

autrement dit, telle que ν est une donnée initiale. On pose le problème variationnel suivant :

$$\lambda_\epsilon = \max_{\nu \in \mathcal{M}_1(M)} \lambda_k(K_\epsilon[\nu]g). \quad (23)$$

Comme l'application $\nu \rightarrow \lambda_k(K_\epsilon[\nu]g)$ est continue et l'espace des mesures de probabilité $\mathcal{M}_1(M)$ est compact on obtient une mesure $\nu_\epsilon \in \mathcal{M}_1(M)$ telle que

$$\lambda_\epsilon = \lambda_k(K_\epsilon[\nu_\epsilon]g).$$

On pose $e^{2u_\epsilon} = K_\epsilon[\nu_\epsilon]$ le facteur conforme associé. Alors, il est facile de montrer que $\{e^{2u_\epsilon}\}$ est une suite maximisante pour $\Lambda_k(M, [g])$.

En utilisant le problème variationnel (23), on obtient une équation d'Euler Lagrange (voir proposition 2) : il existe $\Phi_\epsilon = (\phi_\epsilon^1, \dots, \phi_\epsilon^m)$ une application dont les coordonnées sont des fonctions propres associées à la métrique $e^{2u_\epsilon}g$ et

$$\begin{cases} \text{(i)} & \Delta_g \phi_\epsilon^i = \lambda_\epsilon e^{2u_\epsilon} \phi_\epsilon^i \\ \text{(ii)} & K_\epsilon \left[|\Phi_\epsilon|^2 \right] \geq 1 \\ \text{(iii)} & K_\epsilon \left[|\Phi_\epsilon|^2 \right] = 1, \nu_\epsilon - \text{presque partout} \end{cases} \quad (24)$$

Cette équation est à regarder à côté de (21) car on a presque l'équation des applications harmoniques : on aimerait avoir $|\Phi_\epsilon|^2 = 1$. Ceci est vrai par exemple si le support de la mesure ν_ϵ recouvre toute la surface, car K_ϵ est injectif, et dans ce cas il suffit de connaître le comportement asymptotique d'une suite d'applications harmoniques. Le support de ν_ϵ étant quelconque, nous avons besoin de nouvelles estimées asymptotiques spécifiques au problème.

Etape 2 : Passages à la limite dans la surface

Pour cette étape, nous renvoyons à la section 4.4, page 95. Noter d'abord que

$$e^{2u_\epsilon}dv_g \rightarrow_\star \nu$$

et que l'objectif est de montrer que ν est absolument continue par rapport à dv_g avec une densité \mathcal{C}^∞ et strictement positive, mais qu'on ne peut pas espérer de convergence dans un espace plus régulier pour cette suite. On va donc plutôt tenter de donner des estimées de régularité sur la suite $\{\Phi_\epsilon\}$.

Auparavant, on définit des points de singularité en-dehors desquels on va pouvoir effectuer ces estimées, q_1, \dots, q_s . Comme c'est un peu technique, on laisse leur définition à une lecture précise du Claim 20 dans le chapitre 4. Retenir que ce sont des points au voisinage desquels on a des phénomènes de concentration de plusieurs types. Noter par exemple que les points de concentration de la suite $\{e^{2u_\epsilon}\}$ (c'est à dire les atomes de la mesure limite ν) sont parmi q_1, \dots, q_s . Soit alors

$$M(\rho) = M \setminus \left(\bigcup_{i=1}^s B_g(q_i, \rho) \right)$$

Voici les estimées de régularité de plus en plus fines qu'on peut donner sur la suite $\{\Phi_\epsilon\}$ dans $M(\rho)$ pour tout $\rho > 0$. Elles sont démontrées dans les claims 21 and 22 :

- $\{\Phi_\epsilon\}$ bornée dans $H^1(M(\rho))$:

On sait par construction que les suites $\|\nabla\Phi_\epsilon\|_{L^2(g)}$ et $\|\Phi_\epsilon\|_{L^2(e^{2u_\epsilon}g)}$ sont bornées. Il s'agit ici de montrer qu'on peut aussi borner la suite $\|\Phi_\epsilon\|_{L^2(g)}$. Pour cela, on utilise une inégalité de Poincaré vraie parce que $\{e^{2u_\epsilon}dv_g\}$ est bornée dans $W^{-1,2}(M(\rho))$ grâce en particulier à la non-concentration de cette suite de mesures.

- $\{\Phi_\epsilon\}$ bornée dans $L^\infty(M(\rho))$

Pour cela, on va utiliser (24). L'équation elliptique (i) :

$$\Delta_g\Phi_\epsilon = \lambda_\epsilon e^{2u_\epsilon}\Phi_\epsilon$$

seule n'est pas suffisante pour obtenir un tel résultat. En effet, $\{e^{2u_\epsilon}\}$ est seulement bornée dans L^1 , ce qui n'est pas suffisant pour tirer de meilleures estimées de régularité sur $\{\Phi_\epsilon\}$. Il va aussi falloir utiliser la condition (iii) dans (24), et les résultats de "non-concentration" loin des points q_1, \dots, q_s .

On procède en choisissant une suite de points x_ϵ en lesquels pour ϵ donné $|\Phi_\epsilon|$ atteint son maximum en x_ϵ et en distinguant des cas selon la distance de x_ϵ au support de ν_ϵ . Si x_ϵ et le support de ν_ϵ sont plus éloignés qu'une distance fixée indépendante de ϵ , alors dans un voisinage indépendant de ϵ des points x_ϵ , $\{e^{2u_\epsilon}\}$ est borné dans L^∞ . Comme $\{\Phi_\epsilon\}$ est bornée dans L^2 , on obtient par (i) que $\{\Phi_\epsilon\}$ est bornée L^∞ dans ce voisinage, d'où le résultat dans ce cas.

Si x_ϵ est sur le support de ν_ϵ , on a la condition (iii) : $K_\epsilon[|\Phi_\epsilon|^2] = 1$. On change d'échelle dans la carte exponentielle centrée en x_ϵ :

$$\tilde{\Phi}_\epsilon(z) = \Phi_\epsilon(\sqrt{\epsilon}z)$$

$$e^{2\tilde{u}_\epsilon}(z) = \epsilon e^{2u_\epsilon}(\sqrt{\epsilon}z)$$

pour transformer l'équation (i) en

$$\Delta_g\tilde{\Phi}_\epsilon = \lambda_\epsilon\tilde{\Phi}_\epsilon e^{2\tilde{u}_\epsilon}.$$

(iii) donne que $\{\tilde{\Phi}_\epsilon\}$ est bornée dans L^2 dans un voisinage de 0 et on a $e^{2\tilde{u}_\epsilon}$ bornée dans L^∞ . Ceci implique que $\{\tilde{\Phi}_\epsilon\}$ est bornée dans L^∞ , d'où le résultat dans ce cas.

Si la distance de x_ϵ au support de ν_ϵ tend vers 0, c'est plus complexe, on renvoie à la démonstration.

- Il existe une suite $\beta_\epsilon \rightarrow 0$ lorsque $\epsilon \rightarrow 0$ telle que $|\Phi_\epsilon| \geq 1 - \beta_\epsilon$ uniformément dans $M(\rho)$

On va utiliser ici la condition (ii) dans (24) : $K_\epsilon[|\Phi_\epsilon|^2] \geq 1$, pour démontrer ce résultat uniforme. Noter qu'il n'est pas immédiat car dans (ii), c'est la régularisée de $|\Phi_\epsilon|^2$ qui est plus grande que 1.

- $\left\{\frac{\Phi_\epsilon}{|\Phi_\epsilon|}\right\}$ uniformément équicontinue dans $M(\rho)$

Grâce à ces estimées, on peut passer à la limite : il existe Φ tel que pour tout $\rho > 0$,

$$\frac{\Phi_\epsilon}{|\Phi_\epsilon|} \rightarrow \frac{\Phi}{|\Phi|} \text{ dans } C^0(M(\rho)) \text{ lorsque } \epsilon \rightarrow 0$$

$$\Phi_\epsilon \rightharpoonup \Phi \text{ dans } H^1(M(\rho)) \text{ lorsque } \epsilon \rightarrow 0$$

où Φ satisfait en conséquence de (24)

$$\Delta_g \Phi = \Lambda_k(\Sigma, [g]) \frac{\Phi}{|\Phi|} \nu \quad (25)$$

On obtient une inégalité sur l'énergie de Φ_ϵ dans la surface :

$$\begin{aligned} \lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g &\geq \int_M \frac{|\nabla \Phi|_g^2}{|\Phi|} dv_g \\ &\geq \Lambda_k(M, [g]) m_0 + \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g \end{aligned} \quad (26)$$

où

$$m_0 = \lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} dv_\epsilon .$$

Etape 3 : Construction d'un arbre de bulles

Pour cette étape, on se réfère à la section 4.5, page 106. La suite de mesures $\{e^{2u_\epsilon} dv_g\}$ peut se concentrer. En effet, on peut écrire la limite faible étoile comme

$$e^{2u_\epsilon} dv_g \rightharpoonup^* \nu_0 + \sum_z M_z \delta_z \text{ lorsque } \epsilon \rightarrow 0$$

où ν_0 est la partie sans atome de la limite faible étoile et la somme est prise sur l'ensemble des points z de concentration de la mesure avec des masses associées $M_z > 0$. Notons m_0 la masse de ν_0 . C'est la même définition que m_0 dans l'étape 2.

Le but de cette étape est de construire un arbre qui rend compte des concentrations possibles à toutes les échelles. Pour une suite d'échelles $\alpha^\epsilon \rightarrow 0$ lorsque $\epsilon \rightarrow 0$ donnée et une suite de centres $p^\epsilon \in M$, on définit dans la carte exponentielle centrée en p^ϵ

$$e^{2\hat{u}^\epsilon(z)} = (\alpha^\epsilon)^2 e^{2u_\epsilon(\alpha^\epsilon z)} .$$

On dit qu'on a une bulle associée à $\{\alpha^\epsilon, p^\epsilon\}$ si la partie sans atome de la limite faible étoile dans \mathbb{R}^2 de la suite $\{e^{2\hat{u}^\epsilon} dz\}$ est non nulle. La masse de cette partie sans atome est appelée la masse de la bulle.

On dit que les deux bulles associées à $\{\alpha_1^\epsilon, p_1^\epsilon\}$ et à $\{\alpha_2^\epsilon, p_2^\epsilon\}$ sont disjointes si

$$\frac{d_g(p_1^\epsilon, p_2^\epsilon)}{\alpha_1^\epsilon + \alpha_2^\epsilon} + \frac{\alpha_1^\epsilon}{\alpha_2^\epsilon} + \frac{\alpha_2^\epsilon}{\alpha_1^\epsilon} \rightarrow +\infty \text{ lorsque } \epsilon \rightarrow +\infty .$$

Dans cette étape, il s'agit de démontrer qu'il existe un nombre fini de bulles deux à deux disjointes telles que la somme de leur masse avec m_0 vaut 1, la masse totale. C'est énoncé dans la proposition 3. Notons $i \in \{1, \dots, N\}$ l'indice de la bulle associée aux suites $\{\alpha_i^\epsilon, p_i^\epsilon\}$ et m_i sa masse de sorte que pour $i \neq j$, les bulles d'indice i et j sont disjointes et

$$\sum_{i=0}^N m_i = 1 . \quad (27)$$

Notons pour $i \in \{1, \dots, N\}$, dans la carte exponentielle centrée en p_i^ϵ

$$e^{2\hat{u}_i^\epsilon(z)} = (\alpha_i^\epsilon)^2 e^{2u_\epsilon(\alpha_i^\epsilon z)} ,$$

$$\hat{\Phi}_i^\epsilon(z) = \Phi^\epsilon(\alpha_i^\epsilon z) .$$

Alors, les conditions dans (24) peuvent s'écrire à l'échelle de la bulle. En particulier, (i) s'écrit

$$\Delta \hat{\Phi}_i^\epsilon = \lambda_\epsilon e^{2\hat{u}_i^\epsilon} \hat{\Phi}_i^\epsilon \quad (28)$$

Etape 4 : Passages à la limite dans les bulles

Pour cette étape, consulter la section 4.6, page 116. Noter d'abord que pour $i \in \{1, \dots, N\}$

$$e^{2\hat{u}_i^\epsilon} dz \rightharpoonup_\star \nu_i \text{ lorsque } \epsilon \rightarrow 0$$

et que l'objectif est de montrer que pour $i \in \{1, \dots, N\}$, ν_i est absolument continue par rapport à dz avec une densité \mathcal{C}^∞ et strictement positive. On distingue deux cas selon la vitesse de convergence vers 0 de la suite $\{\alpha_i^\epsilon\}$.

Lorsque $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} \rightarrow +\infty$

Dans ce cas, on ne peut pas obtenir de meilleure convergence pour $\{e^{2\hat{u}_i^\epsilon} dz\}$ qu'une limite faible étoile au sens des mesures et on procède comme dans l'étape 2. On peut effectuer des estimées de régularité de plus en plus fines sur la suite $\hat{\Phi}_i^\epsilon$ de façon à passer à la limite, en dehors de points de singularité $q_1^i, \dots, q_{s_i}^i$. On pose

$$D(\rho) = \mathbb{D}_{\frac{1}{\rho}} \setminus \left(\bigcup_{j=1}^{s_i} \mathbb{D}_\rho(q_j^i) \right) .$$

Il existe $\hat{\Phi}_i$ tel que pour tout $\rho > 0$,

$$\frac{\hat{\Phi}_i^\epsilon}{|\hat{\Phi}_i^\epsilon|} \rightarrow \frac{\hat{\Phi}_i}{|\hat{\Phi}_i|} \text{ dans } \mathcal{C}^0(D(\rho)) \text{ lorsque } \epsilon \rightarrow 0$$

$$\hat{\Phi}_i^\epsilon \rightharpoonup \hat{\Phi}_i \text{ dans } H^1(D(\rho)) \text{ lorsque } \epsilon \rightarrow 0$$

où $\hat{\Phi}_i$ satisfait en conséquence de (28)

$$\Delta \hat{\Phi}_i = \Lambda_k(\Sigma, [g]) \frac{\hat{\Phi}_i}{|\hat{\Phi}_i|} \nu_i \quad (29)$$

On obtient une inégalité sur l'énergie de $\hat{\Phi}_i^\epsilon$ dans la bulle d'indice i

$$\begin{aligned} \lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{D(\rho)} |\nabla \hat{\Phi}_i^\epsilon| dz &\geq \int_{\mathbb{R}^2} \frac{|\nabla \hat{\Phi}_i|}{|\hat{\Phi}_i|} dz \\ &\geq \Lambda_k(M, [g]) m_i + \int_{\mathbb{R}^2} \frac{|\nabla |\hat{\Phi}_i||}{|\hat{\Phi}_i|} dz \end{aligned} \quad (30)$$

où m_i est défini dans l'étape 3 comme la masse de la bulle d'indice i et vaut en particulier

$$m_i = \lim_{\rho \rightarrow 0} \int_{D(\rho)} d\nu_i .$$

Lorsque $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} = O(1)$

Dans ce cas, on est à la bonne échelle pour que $\{e^{2\hat{u}_i^\epsilon}\}$ converge dans $\mathcal{C}^0(\mathbb{D}_R)$. Il existe \hat{u}_i tel que pour tout $R > 0$,

$$e^{2\hat{u}_i^\epsilon} \rightarrow e^{2\hat{u}_i} \text{ dans } \mathcal{C}^0(\mathbb{D}_R) \text{ lorsque } \epsilon \rightarrow 0$$

En utilisant l'équation elliptique (28) et (iii) dans (24), il existe $\hat{\Phi}_i$ tel que pour tout $R > 0$,

$$\hat{\Phi}_i^\epsilon \rightarrow \hat{\Phi}_i \text{ dans } \mathcal{C}^1(\mathbb{D}_R) \text{ lorsque } \epsilon \rightarrow 0.$$

Il reste à donner une inégalité sur l'énergie de $\hat{\Phi}_i^\epsilon$ dans la bulle d'indice i

$$\lim_{R \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{D}_R} |\nabla \hat{\Phi}_i^\epsilon| dz \geq \int_{\mathbb{R}^2} |\nabla \hat{\Phi}_i|^2 dz \geq \Lambda_k(M, [g]) \int_{\mathbb{R}^2} dv_i \quad (31)$$

Dans les deux cas, lorsque $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} \rightarrow +\infty$ et lorsque $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} = O(1)$, nous avons besoin d'un résultat difficile stipulant qu'on ne perd pas d'énergie dans les cous (voir les claims 28 et 31).

Etape 5 : Utilisation d'un théorème de régularité des applications faiblement harmoniques

Cette étape est donnée dans la section 4.7.1, page 142. En combinant les inégalités (26), (30), (31) et (27), on obtient que ce sont des égalités et que pour i tel que $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} \rightarrow +\infty$,

$$|\hat{\Phi}_i|^2 = 1 \text{ sur } \mathbb{R}^2$$

et que

$$|\Phi|^2 = 1 \text{ sur } M.$$

En calculant $\Delta |\hat{\Phi}_i|^2 = 0$ et $\Delta_g |\Phi|^2 = 0$, on obtient grâce à (29) et (25) l'équation des applications harmoniques

$$\Delta \hat{\Phi}_i = |\hat{\Phi}_i|^2 \hat{\Phi}_i \text{ avec } v_i = \frac{|\hat{\Phi}_i|^2}{\Lambda_k(M, [g])} dz$$

et

$$\Delta_g \Phi = |\Phi|_g^2 \Phi \text{ avec } v_0 = \frac{|\Phi|_g^2}{\Lambda_k(M, [g])} dv_g.$$

Grâce à la régularité des applications faiblement harmoniques d'après Hélein [51], les applications $\hat{\Phi}_i$ et Φ sont \mathcal{C}^∞ et les mesures v_i et v sont absolument continues par rapport à dz et dv_g avec des densités

$$e^{2\hat{u}_i} = \frac{|\hat{\Phi}_i|^2}{\Lambda_k(M, [g])} \text{ et } e^{2u_0} = \frac{|\Phi|_g^2}{\Lambda_k(M, [g])}$$

strictement positives (sauf en un nombre fini de points qui correspondent à des singularités coniques) et \mathcal{C}^∞ .

Pour i tel que $\frac{\alpha_i^\epsilon}{\sqrt{\epsilon}} = O(1)$, on sait déjà que $v_i = e^{2\hat{u}_i} dz$ est absolument continue par rapport à dz avec densité strictement positive et \mathcal{C}^∞ .

Etape 6 : Enlever les points de concentration grâce à l'hypothèse d'inégalité (13) stricte

Pour cette étape, consulter la section 4.7.2, page 145. On a obtenu dans l'étape précédente que les mesures limites v_i dans les bulles sont en fait de la forme $e^{2\hat{u}_i} dz$ où $e^{2\hat{u}_i}$ est une fonction

strictement positive (sauf éventuellement en un nombre fini de points qui correspondent à des singularités coniques) et régulière. On peut transporter ces mesures sur une sphère grâce à la projection stéréographique de pôle $p \in \mathbb{S}^2$, $\sigma : \mathbb{S}^2 \setminus \{p\} \rightarrow \mathbb{R}^2$:

$$\sigma(x) = \frac{x - (x.p)p}{1 - (x.p)}.$$

On pose

$$e^{2\tilde{u}_i^\epsilon} dv_h = \sigma^* \left(e^{2\tilde{u}_i^\epsilon} dz \right)$$

où h est la métrique ronde de la sphère. On construit alors des fonctions test à partir des fonctions propres associées à $e^{2\tilde{u}_i^\epsilon} h$ sur la sphère \mathbb{S}^2 et des fonctions propres associées à $e^{2u_0} g$ sur la surface M . En testant convenablement un espace de $k + 1$ fonctions test sur M , on obtient le cas d'égalité dans l'inégalité (13), ce qui contredit l'existence d'un arbre de bulles et donc l'existence de points de concentration de $\{e^{2u_\epsilon}\}$. En refaisant alors l'étape 2 et l'étape 5, on obtient la métrique maximale régulière voulue.

3 Valeurs propres de Steklov

3.1 Bornes sur les valeurs propres pour des surfaces compactes

Soit Σ une surface compacte orientable avec un bord $\partial\Sigma$ lisse. Notons γ son genre et supposons que le bord de la surface a un nombre fini de composantes connexes noté m . Les deux paramètres (γ, m) caractérisent la topologie des surfaces compactes connexes orientables avec un nombre fini de composantes de bords. Etant donnée une métrique riemannienne g sur Σ , on définit l'opérateur de Dirichlet-Neumann $T : \mathcal{C}^\infty(\partial\Sigma) \rightarrow \mathcal{C}^\infty(\partial\Sigma)$ comme suit : pour $u \in \mathcal{C}^\infty(\partial\Sigma)$, on considère le prolongement harmonique \hat{u} de u sur Σ

$$\begin{cases} \Delta_g \hat{u} = 0, \text{ dans } \Sigma \\ \hat{u} = u, \text{ sur } \partial\Sigma \end{cases}.$$

On pose alors $Tu = \partial_\nu \hat{u}$ où ν est la normale extérieure le long de $\partial\Sigma$. Cet opérateur est autoadjoint et a un spectre discret

$$0 = \sigma_0 < \sigma_1(\Sigma, g) \leq \sigma_2(\Sigma, g) \leq \dots \leq \sigma_k(\Sigma, g) \leq \dots \rightarrow +\infty$$

de valeurs appelées valeurs propres de Steklov comptées avec multiplicité. Ce sont les solutions σ de

$$\begin{cases} \Delta_g u = 0, \text{ dans } \Sigma \\ \partial_\nu u = \sigma u, \text{ sur } \partial\Sigma \end{cases}$$

où u est une fonction non nulle, \mathcal{C}^∞ jusqu'au bord de Σ . Ces valeurs propres sont aussi caractérisées par le problème variationnel suivant ressemblant à (4)

$$\sigma_k(\Sigma, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_{\partial\Sigma} \phi^2 d\sigma_g}, \quad (32)$$

où l'infimum est pris parmi les espaces vectoriels de fonctions E_{k+1} dans $\mathcal{C}^\infty(\Sigma)$ de dimension $k + 1$. Le but est de donner des bornes sur ces fonctionnelles dépendant de la métrique g . Par

invariance par dilatation, nous étudierons plutôt les fonctionnelles $\sigma_k(\Sigma, g)L_g(\partial\Sigma)$ où $L_g(\partial\Sigma)$ est la longueur du bord. La borne inférieure est ici triviale

$$\inf_g \sigma_k(\Sigma, g)L_g(\partial\Sigma) = 0 .$$

La démonstration est la même que pour les variétés compactes dans la proposition 1. On s'intéresse ainsi à la borne suivante :

$$\sigma_k(\gamma, m) = \sup_g \sigma_k(\Sigma, g)L_g(\partial\Sigma) .$$

Girouard et Polterovich [47] ont donné une borne sur cet invariant topologique qui ne dépend que de γ et m :

$$\sigma_k(\gamma, m) \leq 2\pi k(\gamma + m)$$

Ceci montre que cet invariant topologique est toujours fini. Ils généralisent pour $k \geq 2$ un résultat de Fraser et Schoen [38] pour $k = 1$. Très peu de valeurs exactes pour $\sigma_k(\gamma, m)$ sont connues. Weinstock [112] a démontré en 1954 que pour tout $k \in \mathbb{N}$,

$$\sigma_k(0, 1) = 2\pi k \tag{33}$$

et que pour $k = 1$, le cas d'égalité a lieu pour le disque euclidien. Pour $k = 2$, Girouard et Polterovich [44] ont montré que la borne de $\sigma_2(0, 1) = 4\pi$ n'était pas atteinte par une métrique régulière. La valeur exacte de $\sigma_1(0, 2)$ a été trouvée par Fraser et Schoen [41] et la métrique maximale associée vient de la caténoïde critique. Enfin, Fraser et Schoen [41] ont aussi donné la limite de $\sigma_k(0, m)$ lorsque $m \rightarrow +\infty$:

$$\lim_{m \rightarrow +\infty} \sigma_k(0, m) = 4\pi$$

3.2 Existence et régularité de métriques maximales

On a l'inégalité analogue à l'inégalité (20) dans le cas du spectre de Steklov :

$$\sigma_k(\gamma, m) \geq \max_{\substack{i_1 + \dots + i_s = k \\ \forall q, i_q \geq 1 \\ \gamma_1 + \dots + \gamma_s \leq \gamma \\ m_1 + \dots + m_s \leq m \\ \gamma_1 < \gamma \text{ ou } m_1 < m \text{ si } s=1}} \sigma_{i_1}(\gamma_1, m_1) + \dots + \sigma_{i_s}(\gamma_s, m_s) . \tag{34}$$

Ceci conduit à l'énoncé du théorème principal d'existence de métriques maximales régulières pour le spectre de Steklov. Ce résultat est démontré dans le chapitre 6.

Théorème 16 (P.). *Soit Σ une surface de genre γ avec un bord lisse ayant $m \geq 1$ composantes connexes. Soit $k \geq 1$. Si l'inégalité (34) est stricte alors il existe une métrique g , de classe C^∞ sur Σ telle que $\sigma_k(\gamma, m) = \sigma_k(g)L_g(\partial\Sigma)$. De plus, quitte à dilater cette métrique maximale, elle est le tiré en arrière d'une métrique euclidienne par une immersion minimale à bord libre dans la boule unité \mathbb{B}^{n+1} pour un certain n .*

Ce résultat avait déjà été démontré par Fraser et Schoen [41] pour la première valeur propre $k = 1$ pour $\gamma = 1$ et pour tout m . Dans ce cas, la condition de l'inégalité (34) stricte

s'écrit $\sigma_1(0, m) > \sigma_1(0, m - 1)$. Ils ont aussi démontré que cette condition est vraie pour tout m de sorte qu'il existe une métrique maximale régulière pour $\sigma_1(0, m)$ pour tout $m \geq 1$.

Noter que l'hypothèse disant que (34) est stricte est nécessaire pour avoir un théorème d'existence. On sait par exemple que par Girouard-Polterovich [44], $\sigma_2(0, 1)$ n'est pas atteinte par une métrique maximale régulière. On remarque que dans ce cas, on a $\sigma_2(0, 1) = 2\sigma_1(0, 1)$ par (33) de sorte que (34) est non stricte.

Même dans le cas $k = 1$, la preuve du théorème 16 diffère de celle de Fraser-Schoen [41]. Pour les valeurs propres plus grandes, par rapport à la première, on doit traiter des phénomènes de bulles et les analyser finement afin de les éliminer grâce à l'hypothèse d'inégalité (34) stricte. Le point de départ de la preuve vient de la remarque suivante : c'est plus simple, même si ce n'est pas facile, de maximiser une valeur propre de Steklov parmi des métriques dans une classe conforme donnée car tout dépend d'une seule fonction. On peut alors sélectionner une suite maximisante de métriques pour $\sigma_k(\gamma, m)$ qui sont maximales dans leur propre classe conforme. Ces maximiseurs viennent avec une application harmonique à bord libre de Σ dans une boule euclidienne et la preuve du théorème 16 repose sur une analyse asymptotique de ces applications harmoniques lorsque la classe conforme dégénère. Des résultats de quantification de ces applications harmoniques à bord libre ont été donnés dans Laurain-Petrides [70].

Ainsi, on introduit l'invariant conforme

$$\sigma_k(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \sigma_k(\tilde{g}) L_{\tilde{g}}(\partial \Sigma)$$

pour toute surface riemannienne compacte (Σ, g) à bord C^∞ non vide où $[g]$ désigne la classe conforme de g . Alors, si Σ est de genre γ avec m composantes de bords, on a

$$\sigma_k(\gamma, m) = \sup_{[g]} \sigma_k(\Sigma, [g]) .$$

Une fois de plus, on a l'inégalité analogue à (13) pour Steklov :

$$\sigma_k(\Sigma, [g]) \geq \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} \left(\sigma_{k-j}(\Sigma, [g]) + \sum_{q=1}^s \sigma_{i_q}(\mathbb{D}, [\zeta]) \right) . \quad (35)$$

Noter que grâce à (33), cette inégalité s'écrit

$$\sigma_k(\Sigma, [g]) \geq \max_{1 \leq j \leq k} (\sigma_{k-j}(\Sigma, [g]) + 2\pi j)$$

mais pour une raison qui deviendra claire dans la démonstration, on préfère garder la forme (35). On obtient alors un résultat d'existence démontré dans le chapitre 6.

Théorème 17 (P). *Soit (Σ, g) une surface riemannienne compacte à bord C^∞ non vide. Alors, si (35) est stricte, il existe une métrique maximale régulière $\tilde{g} \in [g]$ telle que $\sigma_k(\Sigma, [g]) = \sigma_k(\Sigma, \tilde{g}) L_{\tilde{g}}(\partial \Sigma)$.*

Noter que par (33) et (35), la condition d'inégalité stricte du théorème serait une conséquence de

$$\sigma_k(\Sigma, [g]) > \sigma_{k-1}(\Sigma, [g]) + 2\pi .$$

Si le théorème s'applique, une métrique maximale \tilde{g} pour $\sigma_k(\Sigma, [g])$ existe et le facteur conforme associé à g pour la métrique \tilde{g} est $\Phi \cdot \partial_\nu \Phi$ sur $\partial\Sigma$, où Φ est une application harmonique de Σ dans \mathbb{B}^{n+1} avec un bord libre, application dont les coordonnées sont des fonctions propres associées à la k -ème valeur propre de Steklov. Une telle application prend ses valeurs dans une boule euclidienne, est harmonique dans Σ et satisfait $|\Phi| = 1$ et $\partial_\nu \Phi$ orthogonal à $T_\Phi S^n$ sur le bord de Σ .

3.3 Quantification des applications harmoniques à bord libre

Dans cette section, on introduit les applications harmoniques à bord libre car ce sont les applications qui apparaissent naturellement comme points critiques des valeurs propres de Steklov dans une classe conforme fixée comme l'ont noté Fraser et Schoen [39]. C'est analogue au rôle que jouent les applications harmoniques à valeurs dans S^n comme points critiques des valeurs propres de Laplace dans une classe conforme fixée comme on l'a vu dans la section 2.3. On discute ici d'un théorème de régularité des applications faiblement harmoniques à bord libre (Théorème 18), utile pour démontrer le Théorème 17, de même que la régularité des applications faiblement harmoniques à valeurs dans S^n permettait de démontrer le Théorème 14 (voir section 2.3, étape 5). On énonce aussi un théorème de quantification des applications harmoniques à bord libre (Théorème 19) utile pour démontrer le Théorème 16 à partir du Théorème 17. Ces résultats de régularité et de quantification donnent une réponse claire aux questions de Fraser et Schoen dans [39].

Soit (M, g) une surface riemannienne avec un bord non vide à m composantes connexes. Fixons $n \geq 2$. Soit \mathbb{B}^{n+1} la boule unité de \mathbb{R}^{n+1} . On dit qu'une application $u : (M, g) \rightarrow \mathbb{B}^{n+1}$ est une application harmonique à bord libre si elle est harmonique, de classe C^∞ jusqu'au bord, $u(\partial M) \in S^n$ et $\partial_\nu u$ est parallèle à u (ou $\partial_\nu u \perp T_u S^n$). L'énergie d'une telle application vaut

$$E(u) = \int_M |\nabla u|_g^2 dv_g = \int_{\partial M} u \cdot \partial_\nu u d\sigma_g.$$

Les applications harmoniques à bord libre s'étudient du même point de vue que les applications harmoniques à valeurs dans S^n , qui sont les points critiques de l'énergie E sous la contrainte $|u|^2 = 1$ sur la surface. La différence porte sur la contrainte : c'est $|u|^2 = 1$ seulement sur le bord. Noter que si une application harmonique à bord libre est conforme, c'est une immersion minimale à bord libre, c'est à dire qu'elle vérifie en plus que la courbure moyenne de l'immersion est nulle.

La restriction au bord de la surface Riemannienne (M, g) d'une application harmonique à bord libre généralise de façon naturelle les applications $\frac{1}{2}$ -harmoniques sur la droite réelle à valeurs dans S^n . Ce sont les applications $u : \mathbb{R} \rightarrow S^n$ telles que $\Delta^{\frac{1}{2}} u$ est parallèle à u . En effet, le prolongement harmonique sur \mathbb{R}_+^2 d'une telle application correspond à une application harmonique à bord libre sur le disque euclidien (\mathbb{D}, ξ) si on identifie \mathbb{R}_+^2 et \mathbb{D} de manière naturelle via l'application $f : \mathbb{R}_+^2 \rightarrow \mathbb{D}$ définie par $f(z) = \frac{z-i}{z+i}$. Les applications $\frac{1}{2}$ -harmoniques ont été étudiées par Da Lio et Rivière [24][25]. Selon le contexte, d'autres généralisations des applications $\frac{1}{2}$ -harmoniques ont été données comme par exemple dans un papier de Millot et Sire [79].

Les applications harmoniques à bord libre ont aussi une définition au sens faible. Une application $u \in H^1(M, \mathbb{B}^{n+1})$ est dite faiblement harmonique à bord libre si $u(x) \in S^n$ pour presque tout $x \in \partial M$ et si pour tout $v \in L^\infty \cap H^1(M, \mathbb{R}^{n+1})$ avec $v(x) \in T_{u(x)} S^n$ pour presque

tout $x \in \partial M$,

$$\int_M \langle \nabla u, \nabla v \rangle_g dv_g = 0.$$

Une telle application est un point critique de l'énergie E par rapport aux variations $u_t = \pi(u + tv)$ pour tout $v \in L^\infty \cap H^1(M, \mathbb{R}^{n+1})$ avec $v(x) \in T_{u(x)}\mathbb{S}^n$ pour presque tout $x \in \partial M$, où pour $z \in \mathbb{R}^{n+1}$, $\pi(z)$ est la projection de z sur \mathbb{B}^{n+1} . On peut alors montrer que :

Théorème 18 (Scheven [102]). *Une application faiblement harmonique à bord libre $u : \mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$ est toujours de classe C^∞ jusqu'au bord et donc est une application harmonique à bord libre au sens classique.*

Ce résultat a été prouvé par Scheven [102] dans un contexte plus général encore. Ici, on obtient l'analogie du résultat de Hélein [51] pour les applications harmoniques à valeurs dans \mathbb{S}^n . En fait, nous démontrons dans [70] un résultat d' ϵ -régularité plus général qu'un résultat de régularité qui permet de démontrer le théorème suivant dans le chapitre 5.

Théorème 19 (Laurain, P. [70]). *Soit $u_\alpha : (M, g) \rightarrow \mathbb{B}^{n+1}$ une suite d'applications harmoniques à bord libre, c'est à dire $u_\alpha(\partial M) \subset \mathbb{S}^n$ et u_α est parallèle à $\partial_\nu u_\alpha$ tel que*

$$\limsup_{\alpha \rightarrow +\infty} \int_M |\nabla u_\alpha|_g^2 dv_g < +\infty.$$

Alors il existe une application harmonique à bord libre $u_\infty : M \rightarrow \mathbb{B}^{n+1}$ et

- $\omega^1, \dots, \omega^l$ une famille d'applications $\frac{1}{2}$ -harmoniques $\mathbb{R} \rightarrow \mathbb{S}^n$
- $a_\alpha^1, \dots, a_\alpha^l$ une famille de suites de points de ∂M convergeant respectivement vers $a_\infty^1, \dots, a_\infty^l$
- $\lambda_\alpha^1, \dots, \lambda_\alpha^l$ une suite de nombres strictement positifs qui convergent vers 0

tels que quitte à extraire une sous-suite

$$u_\alpha \rightarrow u_\infty \text{ dans } C_c^\infty(M \setminus \{a_\infty^1, \dots, a_\infty^l\}),$$

et

$$\int_{\partial M} R_\alpha \partial_\nu R_\alpha \rightarrow 0$$

où

$$R_\alpha = u_\alpha - u_\infty - \sum_{i=1}^l \omega^i \left(\frac{\cdot - a_\alpha^i}{\lambda_\alpha^i} \right)$$

en identifiant ∂M avec m copies de $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$:

$$\partial M = \bigcup_{j=1}^m C_j$$

avec pour tout $1 \leq i \leq l$, $a_\infty^i \in C_j \setminus \{\infty\}$ pour un certain j .

En particulier, dans l'espace des mesures sur le bord, nous avons la convergence

$$u_m \partial_\nu u_m d\sigma_g \rightharpoonup_\star u_\infty \partial_\nu u_\infty d\sigma_g + \sum_{i=1}^l e_i \delta_{a_\infty^i},$$

où e_i est l'énergie du prolongement harmonique de ω^i sur \mathbb{R}_+^2 qui est harmonique à bord libre et qu'on note aussi $\omega^i : \mathbb{R}_+^2 \rightarrow \mathbb{B}^{n+1}$. C'est aussi l'énergie de l'application $\omega^i \circ f^{-1}$

harmonique à bord libre sur le disque, où $f : \mathbb{R}_+^2 \rightarrow \mathbb{D}$ est définie par $f(z) = \frac{z-i}{z+i}$. Ainsi, ω^i est automatiquement une application conforme d'après Fraser et Schoen [40], et $\omega^i \circ f^{-1}(\mathbb{D})$ est un disque plan équatorial et l'énergie d'une telle application satisfait

$$e_i = E(\omega^i) = \int_{\mathbb{R} \times \{0\}} \omega^i(-\partial_t \omega^i) ds \in 2\pi\mathbb{N}.$$

Ce résultat est donc bien un résultat de quantification, analogue à ceux de Sacks-Uhlenbeck [100], Parker [88] pour les applications harmoniques à valeurs dans S^n ou Laurain-Rivière [71] pour des équations similaires. Ce résultat de quantification avait été démontré par Da Lio sur le disque [24] par une autre méthode, en écrivant l'énergie d'une application $\frac{1}{2}$ -harmonique comme

$$E(u) = \int_{\mathbb{R}} \left| \Delta^{\frac{1}{4}} u \right|^2 dx$$

et en exprimant ce qu'est un point critique de cette énergie.

Chapitre 1

Maximisation de la deuxième valeur propre conforme sur des sphères

Dans ce chapitre, nous donnons une borne supérieure sur la deuxième valeur propre des sphères de dimension n dans la classe conforme de la sphère ronde. Cette borne supérieure a lieu en toute dimension et est asymptotiquement optimale quand la dimension augmente.

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1.1 Introduction

Given (M, g) a smooth compact Riemannian manifold (without boundary), the spectrum of the Laplacian $\Delta_g = -\operatorname{div}_g(\nabla)$ is a discrete sequence of eigenvalues

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \cdots \leq \lambda_k(M, g) \leq \dots$$

which goes to $+\infty$ as $k \rightarrow +\infty$. The eigenfunctions associated to the simple eigenvalue $\lambda_0 = 0$ are the constant functions. A natural, and often addressed, question is to get estimates on the eigenvalues thanks to some geometric assumptions. In this paper, we discuss maximisation of eigenvalues for metrics in a given conformal class with fixed volume. We focus on the case of the standard sphere.

We let \mathbf{S}^n be the unit sphere of \mathbf{R}^{n+1} for $n \geq 2$. If g is a metric on \mathbf{S}^n , we are interested in the scale invariant quantity

$$\Lambda_{n,k}(g) = \lambda_k(\mathbf{S}^n, g) \operatorname{Vol}_g(\mathbf{S}^n)^{\frac{2}{n}}$$

In dimension 2, we can maximize $\Lambda_{2,k}$ on regular metrics. An inequality has been proved for $k = 1$ by Hersch [54] :

$$\Lambda_{2,1}(g) \leq 8\pi$$

with equality iff g is the round metric. He followed the proof of the maximization by Szegő [107] of the first non zero Neumann eigenvalue for planar domains, attained by discs. Nadi-rashvili found an optimal maximization for $k = 2$. He proved in [84] that

$$\Lambda_{2,2}(g) < 16\pi$$

where the supremum is attained in the degenerate case of the union of two identical spheres. His idea was used later in [46] to show that among simply connected planar domains, the second non zero Neumann eigenvalue is maximal in the degenerate case of two discs of the same area.

If we look for an analogous inequality in dimension $n \geq 3$, we have to restrict our attention to some classes of metrics since $\Lambda_{n,k}$ is not bounded on the set of regular metrics (see [20]). It is natural, as suggested in [33] and [21], to consider the set of metrics in some conformal class. Indeed, in any given conformal class, $\Lambda_{n,k}(g)$ admits some upper bound (see [67]). Thus we define the conformal spectrum of $(\mathbf{S}^n, [g_0])$, where $[g_0]$ is the class of metrics conformal to the round metric g_0 , by

$$\lambda_k^c(\mathbf{S}^n, [g_0]) = \sup_{g \in [g_0]} \Lambda_{n,k}(g)$$

The theorem of Hersch was generalized in this framework in [33]. We have that

$$\lambda_1^c(\mathbf{S}^n, [g_0]) = n\sigma_n^{\frac{2}{n}}$$

where σ_n is the volume of the unit n -dimensional sphere. We know almost nothing about $\lambda_k^c(\mathbf{S}^n, [g_0])$ for $k \geq 2$. A lower bound was obtained by a method of conformal surgery in [21]. For all k , we have that

$$\lambda_k^c(\mathbf{S}^n, [g_0]) \geq n(k\sigma_n)^{\frac{2}{n}}.$$

Nadirashvili, Girouard and Polterovich conjectured in [46] that this inequality is an equality in all dimensions for $k = 2$, where the supremum is attained for the union of two identical spheres :

Conjecture ([46]) : *for any metric $g \in [g_0]$,*

$$\lambda_2(\mathbf{S}^n, g) \text{Vol}_g(\mathbf{S}^n)^{\frac{2}{n}} < n(2\sigma_n)^{\frac{2}{n}}.$$

In the way to this conjecture, the following theorem gives an "asymptotically sharp" upper bound :

Theorem 1. *Let $n \geq 2$ and $g \in [g_0]$ a metric on S^n conformal to the round metric. Then*

$$\lambda_2(\mathbf{S}^n, g) \text{Vol}_g(\mathbf{S}^n)^{\frac{2}{n}} < K_n n(2\sigma_n)^{\frac{2}{n}}$$

where K_n is a constant independant of $g \in [g_0]$ given by

$$K_n = \frac{n+1}{n} \left(\frac{\Gamma(n)\Gamma(\frac{n+1}{2})}{\Gamma(n+\frac{1}{2})\Gamma(\frac{n}{2})} \right)^{\frac{2}{n}}.$$

Note that $K_2 = 1$, that $1 < K_n \leq 1.04$ for all $n \geq 3$ and that $\lim_{n \rightarrow \infty} K_n = 1$. The theorem is sharp in dimension 2 and was in fact already proved by Nadirashvili in [84]. In [46], Girouard, Nadirashvili and Polterovich established this inequality in odd dimensions.

We prove in this paper this theorem in all dimensions, unifying the previous proofs in dimension $n = 2$ and in odd dimensions and by the way extending it. The starting point of the proof is a construction, described in section 1.2 below, initiated by Nadirashvili [84] and used by Girouard, Nadirashvili and Polterovich [46] in odd dimension. However, our use of this construction differs from that of these two papers : we use the min-max characterisation of the second eigenvalue up to the end of the proof (see section 1.4), capitalizing on a new topological fact proved in section 1.3.

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1.2 Construction of test functions

In this section, we describe the construction of Nadirashvili [84] (see also [46]) which is at the basis of our theorem as well as of the previous results. Let g be a metric on \mathbf{S}^n conformal to g_0 of volume 1. We denote by dv_g the measure associated to g . We shall use in this paper the min-max characterization of the second eigenvalue of the Laplacian which tells us in particular that

$$\lambda_2(\mathbf{S}^n, g) \leq \sup_{u \in E \setminus \{0\}} \frac{\int_{\mathbf{S}^n} |\nabla_g u|_g^2 dv_g}{\int_{\mathbf{S}^n} u^2 dv_g} \quad (1.1)$$

for all 2-dimensional subspaces E of functions in $H^1(\mathbf{S}^n)$ with mean value 0. The aim is to find a suitable space E of test-functions such that (1.1) gives the estimate of the theorem.

On (\mathbf{S}^n, g_0) , the eigenspace associated to $\lambda_1(\mathbf{S}^n, g_0)$ has dimension $n + 1$: it is the set of linear forms of \mathbf{R}^{n+1} written $X_s = (s, \cdot)$ for $s \in \mathbf{R}^{n+1}$. We will build E with these functions, and as Hersch did for $\lambda_1(\mathbf{S}^n, g)$, we proceed to a renormalisation of measures in order to keep the orthogonality to constants. For $\xi \in \mathbf{B}^{n+1}$, we let $d_\xi : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{n+1}$ be defined by

$$d_\xi(x) = \frac{(1 - |\xi|^2)x + (1 + 2\xi \cdot x + |x|^2)\xi}{1 + 2\xi \cdot x + |x|^2 |\xi|^2}$$

which is a conformal transformation when restricted to the unit sphere.

We say that d_ξ renormalizes a finite measure dv on the n -sphere if

$$\forall s \in \mathbf{S}^n, \int_{\mathbf{S}^n} X_s \circ d_\xi dv = 0.$$

The Hersch lemma says that for all finite measures dv , such a ξ exists. Moreover it is unique and depends continuously on dv (the set of finite measures is considered as the topological dual of the continuous bounded functions) as proved in [46], Proposition 4.1.5. We call ξ the renormalization point of dv .

We also define families of measures parametrized by the set of caps of \mathbf{S}^n , denoted by \mathcal{C} :

$$a_{0,p} = \{x \in \mathbf{S}^n; x \cdot p > 0\} \quad a_{r,p} = d_{rp}(a_{0,p}) \quad (r, p) \in (-1, 1) \times \mathbf{S}^n$$

We denote by $d\mu_a$ the "lift" of the measure dv_g by the cap $a \in \mathcal{C}$:

$$d\mu_a = \begin{cases} dv_g + (\tau_a)^* dv_g & \text{on } a \\ 0 & \text{on } a^* \end{cases}$$

where $a^* = \mathbf{S}^n \setminus \bar{a}$ and τ_a is the conformal reflection with respect to the boundary circle of a , that is

$$\tau_{a,r,p} = d_{rp} \circ R_p \circ d_{-rp}$$

where

$$R_p(x) = x - 2(p, x)p$$

is the reflection of \mathbf{R}^{n+1} with respect to the hyperplane orthogonal to p . Let $\xi(a)$ be the renormalization point of $d\mu_a$. We set $dv_a = (d_{\xi(a)})_* d\mu_a$. Thanks to this family of measures, we can define a new family of test functions orthogonal to the constants :

$$u_a^s = \begin{cases} X_s \circ d_{\xi(a)} & \text{on } a \\ X_s \circ d_{\xi(a)} \circ \tau_a & \text{on } a^* \end{cases}$$

By a Hölder inequality, the numerator of the Rayleigh quotient is less than a conformal invariant.

$$\begin{aligned} \int_{\mathbf{S}^n} |\nabla_g u_a^s|_g^2 dv_g &< \left(\int_{\mathbf{S}^n} |\nabla_g u_a^s|_g^n dv_g \right)^{\frac{2}{n}} \\ &= \left(2 \int_{d_{\xi(a)}(a)} |\nabla_g X_s|_g^n dv_g \right)^{\frac{2}{n}} \\ &< \left(2 \int_{\mathbf{S}^n} |\nabla_{g_0} X_s|_{g_0}^n dv_{g_0} \right)^{\frac{2}{n}} \end{aligned} \tag{1.2}$$

Let us define the multiplicity of a finite measure :

definition. *The multiplicity of a finite measure dv on \mathbf{S}^n is the dimension of the eigenspace W associated to the maximal eigenvalue of the quadratic form :*

$$Q(s) = \int_{\mathbf{S}^n} X_s^2 dv$$

We say that dv is multiple if its multiplicity is greater than or equal to 2. Otherwise, we say that dv is simple.

As was noticed in [46], we know that if dv_g is multiple, then we can choose $E = \{X_s; s \in W\}$ in (1.1) to get that $\lambda_2(\mathbf{S}^n, g) \leq n(2\sigma_n)^{\frac{2}{n}}$. We also know that if there is a cap $a \in \mathcal{C}$ such that dv_a is multiple, $\lambda_2(\mathbf{S}^n, g) < K_n n(2\sigma_n)^{\frac{2}{n}}$ using the space of test functions $E = \{u_a^s; s \in W\}$ in (1.1). In this case, the theorem would be proved. In [46], it was proved that there necessarily exists such a multiple measure in odd dimensions (see below).

Let us now assume that all measures dv_g and dv_a , for $a \in \mathcal{C}$, are simple. Up to a renormalisation and a rotation, we may assume that

$$\forall t \in \mathbf{S}^n, \int_{\mathbf{S}^n} X_t dv_g = 0$$

and that

$$\forall t \in \mathbf{S}^n \setminus [e_1], \int_{\mathbf{S}^n} X_t^2 dv_g < \int_{\mathbf{S}^n} X_{e_1}^2 dv_g.$$

We denote by $[s(a)]$ the unique direction of maximization of the quadratic form associated to dv_a . With the parametrization $(r, p) \in (-1, 1) \times \mathbf{S}^n$ of \mathcal{C} , the maps $\xi : \mathcal{C} \rightarrow \mathbf{B}^{n+1}$ and $[s] : \mathcal{C} \rightarrow \mathbf{RP}^n$ are continuous. Moreover, one may prove that if $r \rightarrow -1$, that is $a \rightarrow \mathbf{S}^n$, we have :

$$\lim_{a \rightarrow \mathbf{S}^n} \xi(a) = 0 \qquad \lim_{a \rightarrow \mathbf{S}^n} [s(a)] = [e_1] \tag{1.3}$$

1.3 Properties of the lift of the maximal direction

Let us study the maps ζ and $[s]$ at the light of the links between a cap $a \in \mathcal{C}$ and its symmetrical cap $a^* = \mathbf{S}^n \setminus \bar{a}$. With the parameter $(r, p) \in (-1, 1) \times \mathbf{S}^n$, notice that $a_{r,p}^* = a_{-r,-p}$.

Claim 1. For $a \in \mathcal{C}$, we write $\zeta^* = \zeta(a^*)$, $[s^*] = [s(a^*)]$. Then

$$-\zeta^* = \tau_a(-\zeta) \quad \text{and} \quad [s^*] = R_a[s]$$

where $R_a = d_{\zeta^*(a)} \circ \tau_a \circ d_{-\zeta(a)}$ is an orthogonal map.

Proof.

We set $\eta = -\tau_a(-\zeta)$. Let $t \in \mathbf{S}^n$, then

$$\int_{\mathbf{S}^n} X_t \circ d_\eta d\mu_{a^*} = \int_{\mathbf{S}^n} X_t \circ d_\eta \circ \tau_a d\mu_a.$$

One can check that $d\mu_{a^*} = (\tau_a)^* d\mu_a$. The map $R_a = d_\eta \circ \tau_a \circ d_{-\zeta(a)}$ is orthogonal because it is a Möbius transformation of the unit ball preserving the origin ([4], Theorem 3.4.1). Thus we have that

$$\int_{\mathbf{S}^n} X_t \circ d_\eta d\mu_{a^*} = \int_{\mathbf{S}^n} X_t \circ R_a \circ d_\zeta d\mu_a = \int_{\mathbf{S}^n} X_{R_a^{-1}(t)} \circ d_\zeta d\mu_a = 0.$$

This is true for all $t \in \mathbf{S}^n$, and uniqueness of the renormalization point ensures that $\zeta^* = \eta$.

The same argument with the function $(X_t \circ d_{\zeta^*})^2$ leads to

$$\forall t \in \mathbf{S}^n, \int_{\mathbf{S}^n} (X_t \circ d_{\zeta^*})^2 d\mu_{a^*} = \int_{\mathbf{S}^n} (X_{R_a^{-1}(t)} \circ d_\zeta)^2 d\mu_a$$

and once again, we can conclude by uniqueness of the maximal direction that $[s^*] = R_a[s]$. \diamond

Remark. Thanks to this claim 1, we can prove the theorem in odd dimensions. Indeed, when $r \rightarrow 1$ that is $a \rightarrow \{p\}$, we use (1.3) in order to obtain :

$$\lim_{a \rightarrow \{p\}} R_a = R_p$$

Then, $[s(a)] = R_a^{-1}[s^*(a)] \rightarrow R_p[e_1]$ when $a \rightarrow \{p\}$ by (1.3). Therefore, following [46] in odd dimensions, the map $[s] : [-1, 1] \times \mathbf{S}^n \rightarrow \mathbf{RP}^n$ defines a homotopy between the constant map $[e_1]$ of degree 0 and $\phi(p) = R_p[e_1]$ of degree 4. Thus, there is a contradiction and there exists a multiple measure among $d\nu_g$ and $d\nu_a$ for $a \in \mathcal{C}$.

We do not prove that the assumption that all measures are simple lead to a contradiction. Indeed, it is not clear that in even dimensions, such a configuration can not happen. Instead, we look for suitable test functions like in Nadirashvili's proof in dimension 2 [84]. However, inspired by the method of [46], we use a topological argument to get symmetric properties of the lifts of the maximal directions.

The continuous map $[s] : [-1, 1] \times \mathbf{S}^n \rightarrow \mathbf{RP}^n$ has exactly two continuous lifts because the set $[-1, 1] \times \mathbf{S}^n$ is simply connected. We denote by s the continuous lift such that $s(-1, \cdot) = -e_1$, the other continuous lift is $-s$. Thanks to claim 1,

$$s(-r, -p) = \epsilon(r, p) R_{a_{r,p}} s(r, p)$$

where $\epsilon : [-1, 1] \times \mathbf{S}^n \rightarrow \{\pm 1\}$ is a continuous map. Since $s \neq 0$ and $[-1, 1] \times \mathbf{S}^n$ is connected, ϵ is a constant map.

Claim 2. We have that $\epsilon = -1$. In other words,

$$s(a^*) = -R_a s(a)$$

for all caps a .

Proof.

We assume by contradiction that $\epsilon = 1$. We set $f(p) = s(0, p)$ for $p \in \mathbf{S}^n$. This function f is continuous on the sphere and satisfies

$$\forall p \in \mathbf{S}^n, f(-p) = R_p f(p) \quad (1.4)$$

Indeed, $R_{a_0, p} = R_p$ because $\tau_{a_0, p} = R_p$. Using claim 3 below, we know that such a map f can not have degree 0. However, the map $s : [-1, 0] \times \mathbf{S}^n \rightarrow \mathbf{S}^n$ defines a homotopy between $s_0 = f$ and $s_{-1} = -e_1$ of degree zero. Thus, there is a contradiction. \diamond

We have used the following topology result :

Claim 3. Let $f : \mathbf{S}^n \rightarrow \mathbf{S}^n$ a continuous map which satisfies (1.4). Then, if n is odd, $\deg(f) = 1$ and if n is even, $\deg(f) \in 2\mathbf{Z} + 1$.

Proof.

We first prove the claim for smooth functions which have a property of transversality (step 1) and we show that this case is generic (step 2).

Step 1 - Let $f : \mathbf{S}^n \rightarrow \mathbf{S}^n$ be a smooth function which satisfies (1.4). Let us assume that for all fix point $x \in \mathbf{S}^n$ of f , $T_x f - I : T_x \mathbf{S}^n \rightarrow T_x \mathbf{S}^n$ is an isomorphism. Then, if n is odd, $\deg(f) = 1$ and if n is even, $\deg(f) \in 2\mathbf{Z} + 1$.

Proof of step 1 - Let F be defined by

$$\begin{aligned} F : \mathbf{S}^n \times [-1, 1] &\longrightarrow \mathbf{R}^{n+1} \\ (x, t) &\longmapsto \frac{1}{2}(f(x) - x + t(f(x) + x)) \end{aligned}$$

We notice that if F never vanishes, $\frac{F}{|F|}$ defines a homotopy between f and σ , the antipodal map and $\deg(f) = \deg(\sigma) = (-1)^{n+1}$.

Now, $F(x, t) = 0$ if and only if $t = 0$ and x is a fix point of f and then,

$$\forall (v, t) \in T_x \mathbf{S}^n \times \mathbf{R}, \quad DF(x, 0)(v, t) = \frac{1}{2}(T_x f - I)v + xt.$$

Thus, $DF(x, 0)$ is an isomorphism, and 0 is a regular value. We write $(x_1, 0), \dots, (x_r, 0)$ the regular points of $F^{-1}(0)$. Let's approximate F by its differential in the neighborhood of its zeros. Let $\alpha > 0$ and, set for $1 \leq i \leq r$, $\phi_i : B_{x_i}(\alpha) \rightarrow B_0(\alpha) \subset T_{x_i} \mathbf{S}^n$ the exponential chart at x_i . We obtain for $(x, t) \in B_{x_i}(\alpha) \times (-\alpha, \alpha)$

$$F(x, t) = DF(x_i, 0)(\phi_i(x), t) + R_i(\phi_i(x), t)$$

where $\frac{R_i(v, t)}{|(v, t)|} \rightarrow 0$ when $(v, t) \rightarrow 0$. We write for $x \in \mathbf{S}^n$ that

$$F_t(x) = F(x, t) \quad L_t(x) = \begin{cases} DF(x_i, 0)(\phi_i(x), t) & \text{if } (x, t) \in B_{x_i}(\alpha) \times (-\alpha, \alpha) \\ 0 & \text{otherwise.} \end{cases}$$

We define a cut-off function $0 \leq \psi \leq 1$ such that $\psi = 1$ on $K_1 = \bigcup_{i=1}^r \overline{B_{x_i}(\frac{\alpha}{2})}$ and $\psi = 0$ on $K_2 = \mathbf{S}^n \setminus \bigcup_{i=1}^r B_{x_i}(\alpha)$. We set for $s \in [0, 1]$

$$G_s^t = \frac{s\psi L_t + (1-s\psi)F_t}{|s\psi L_t + (1-s\psi)F_t|}.$$

One may choose $\alpha > 0$ small enough so that G_s^t is well defined for all $t \in (-\alpha, \alpha) \setminus \{0\}$. Then, for $0 < t < \alpha$, G_1^t is homotopic to $G_0^t = \frac{F_t}{|F_t|}$, so to f , and G_1^{-t} is homotopic to σ . We now write, for $t \in (-\alpha, \alpha)$, $g_t = G_1^t$.

Let us look at the behaviour of $g_t = \frac{L_t}{|L_t|}$ in the balls $\overline{B_{x_i}(\frac{\alpha}{2})}$ when $t \rightarrow 0$. We recall that

$$L_t(x) = \frac{1}{2}(T_{x_i}f - I)\phi_i(x) + x_i t.$$

Therefore, the image $I_{x_i}^t = g_t(\overline{B_{x_i}(\frac{\alpha}{2})})$ blows up to the half-sphere $D_{x_i} = \{x \in \mathbf{S}^n; (x, x_i) > 0\}$ when $t \rightarrow 0$.

Thanks to (1.4), x is a fix point of f if and only if $-x$ is a fix point too. Moreover, by differentiating (1.4) at a fix point x , we obtain $T_{-x}f - I = -(T_x f - I)$.

Let's renumber the fix points $x_1, \dots, x_k, -x_1, \dots, -x_k$ (with $r = 2k$), so that x_1, \dots, x_k are in a same half sphere $D_p = \{(x, p) > 0\}$. We choose $\epsilon < \alpha$ small enough so that $\bigcap_{i=1}^k I_{x_i}^\epsilon$ has a non-empty interior I . Then, for $z \in I$, there is a unique point in $g_t^{-1}(z) \cap B_{x_i}(\frac{\alpha}{2})$ for all $0 < t < \epsilon$. Since $g_\epsilon(x) = g_{-\epsilon}(-x)$, if $z \in I$, then $z \in I_{-x_i}^{-\epsilon}$ and $z \notin I_{x_i}^\epsilon \cup I_{x_i}^{-\epsilon}$.

For $1 \leq i \leq k$, let $\{a_i\} = B_{x_i}(\frac{\alpha}{2}) \cap g_\epsilon^{-1}(z)$. Then by definition of degree and homotopy,

$$\deg(f) - \deg(\sigma) = \deg(g_\epsilon) - \deg(g_{-\epsilon}) = \sum_{i=1}^k \text{ind}_{a_i}(g_\epsilon) - \text{ind}_{-a_i}(g_{-\epsilon}) = \sum_{i=1}^k (1 - (-1)^{n+1})\nu_i$$

where $\nu_i = \text{ind}_{a_i}(g_\epsilon) \in \pm 1$. In odd dimensions, $\deg(f) = \deg(\sigma) = 1$ and in even dimensions, $\deg(f) \in 2\mathbf{Z} + 1$. This ends the proof of step 1.

Step 2 - Let $f : \mathbf{S}^n \rightarrow \mathbf{S}^n$ be a continuous map which satisfies (1.4). Then there exists a map, homotopic to f , which satisfies the assumptions of step 1.

Proof of step 2 - Denote by (e_0, e_1, \dots, e_n) the canonical basis of \mathbf{R}^{n+1} and $B_k^\alpha \subset D_{e_k} = \{(x, e_k) > 0\}$ the ball centered at e_k such that $d(B_k^\alpha, D_{-e_k}) = \alpha > 0$. Choose α small enough so that

$$\bigcup_{i=0}^n B_i^{2\alpha} \cup (-B_i^{2\alpha}) = \mathbf{S}^n.$$

Let $\epsilon > 0$. We build by induction maps $g_k : \mathbf{S}^n \rightarrow \mathbf{S}^n$ such that $g_0 = f$ and, for $0 \leq k \leq n$,

- $g_{k+1} = g_k$ on $\mathbf{S}^n \setminus (B_k^\alpha \cup (-B_k^\alpha))$
- g_{k+1} is smooth on $\bigcup_{i=0}^k B_i^{2\alpha} \cup (-B_i^{2\alpha})$
- $\|g_{k+1} - g_k\|_{C^0} < \epsilon$
- g_{k+1} satisfies (1.4).

By density of smooth maps $\mathbf{S}^n \rightarrow \mathbf{R}^{n+1}$, choose h_k such that $\|h_k - g_k\|_{C^0} < \epsilon$. Let $0 \leq \phi \leq 1$ be a smooth cut-off function such that $\phi = 1$ on $B_i^{2\alpha}$ and $\phi = 0$ on $\mathbf{S}^n \setminus B_i^\alpha$. We let g_{k+1} be defined, provided ϵ is small enough, by

$$g_{k+1}(x) = \frac{\phi h_k + (1-\phi)g_k}{|\phi h_k + (1-\phi)g_k|} \text{ and } g_{k+1}(-x) = R_x \circ g_{k+1}(x)$$

for $x \in \overline{D_{e_k}}$. Therefore $g = g_{n+1}$ is smooth, satisfies (1.4) and $\|g - f\|_{C^0} < C\epsilon$. If ϵ is small enough, g is homotopic to f .

Let's now tackle the transversality condition. We write g in the following way

$$g(x) = X(x) + \lambda(x)x$$

where X is a tangent vector field of the sphere and $|X|^2 + \lambda^2 = 1$. Then, g satisfies (1.4) if and only if X and λ are even maps. By differentiating these equalities at a fix point x (with $\lambda(x) = 1$ and $X(x) = 0$), one may find $T_x g - I = T_x X$. Then, $T_x g - I$ is an isomorphism for all fix points x if and only if X is transverse to the zero vector field. Then, one may build by induction, with Sard's theorem in n -dimensional charts on D_{e_k} , smooth tangent vector fields X_k such that $X_0 = X$ and for $0 \leq k \leq n$:

- $X_{k+1} = X_k$ on $\mathbf{S}^n \setminus (B_k^\alpha \cup (-B_k^\alpha))$
- X_{k+1} is transverse to 0 on $\bigcup_{i=0}^k B_i^{2\alpha} \cup (-B_i^{2\alpha})$
- $\|X_{k+1} - X_k\|_{C^0} < \epsilon$
- X_{k+1} is an even map.

Set $\bar{f}(x) = \frac{X_{n+1}(x) + \lambda(x)x}{|X_{n+1}(x)|^2 + \lambda(x)^2}$. If ϵ is small enough, then \bar{f} is well defined, satisfies the assumptions of step 1 and is homotopic to f . This ends the proof of step 2.

These two steps clearly end the proof of the claim. ◇

1.4 Choice of test functions

Thanks to claim 2, one may easily deduce that

$$\forall a \in \mathcal{C}, u_{a^*} = -u_a \quad (1.5)$$

where we have set, for this section $u_a = u_a^{s(a)}$. Let $r \in (-1, 1)$. We look at the space E generated by

$$\phi = X_{e_1} \text{ and } \psi_r = u_{a_{r,e_1}}.$$

One may deduce from the continuity of ξ and s , (1.3) and (1.5), that

Claim 4. *The map $r \in (-1, 1) \mapsto \psi_r \in (L^2(\mathbf{S}^n, g), \|\cdot\|_{L^2})$ is continuous and*

$$\lim_{r \rightarrow -1} \psi_r = -\phi \quad \lim_{r \rightarrow 1} \psi_r = \phi$$

For $(x, y) \in \mathbf{R}^2 \setminus \{0\}$, we set $f_r = x\phi + y\psi_r \in E$. Conformal invariance gives that

$$\frac{\int_{\mathbf{S}^n} |\nabla_g f_r|_g^2 dv_g}{\int_{\mathbf{S}^2} f_r^2 dv_g} = \frac{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}}{\frac{1}{n+1}} \frac{\sigma x^2 + \tau_r y^2 + 2\alpha_r xy}{Ix^2 + J_r y^2 + 2\beta_r xy} := (n+1) \left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}} q(x, y)$$

where we set for $r \in (-1, 1)$

$$\sigma = \frac{\int_{\mathbf{S}^n} |\nabla_g \phi|_g^2 dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}} < 1 \quad \tau_r = \frac{\int_{\mathbf{S}^n} |\nabla_g \psi_r|_g^2 dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}} < 2^{\frac{2}{n}}$$

$$\begin{aligned}\alpha_r &= \frac{\int_{\mathbf{S}^n} g(\nabla_g \psi_r, \nabla_g \phi) dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}} & \beta_r &= (n+1) \int_{\mathbf{S}^n} \phi \psi_r dv_g \\ I &= (n+1) \int_{\mathbf{S}^n} \phi^2 dv_g > 1 & J_r &= (n+1) \int_{\mathbf{S}^n} \psi_r^2 dv_g > 1\end{aligned}$$

By (1.2), $\tau_r < 2^{\frac{2}{n}}$ and by maximality of ϕ and ψ_r , $I > 1$ and $J_r > 1$.

The value $(n+1)2^{\frac{2}{n}} \left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}$ which also appears in (1.2), is independant of the metric $g \in [g_0]$ thanks to conformal invariance. The quotient K_n given in the theorem compares this value with the constant of the conjecture $n(2\sigma_n)^{\frac{2}{n}}$.

$$\begin{aligned}K_n &:= \frac{(n+1)2^{\frac{2}{n}} \left(\int_{\mathbf{S}^n} |\nabla_{g_0} \phi|_{g_0}^n dv_{g_0} \right)^{\frac{2}{n}}}{n(2\sigma_n)^{\frac{2}{n}}} \\ &= \frac{n+1}{n} \left(\frac{1}{\sigma_n} \int_{\mathbf{S}^n} (1 - X_{e_1}^2) dv_{g_0} \right)^{\frac{2}{n}} \\ &= \frac{n+1}{n} \left(\frac{\sigma_{n-1}}{\sigma_n} \int_0^\pi (\sin \theta)^{2n-1} d\theta \right)^{\frac{2}{n}}\end{aligned}\tag{1.6}$$

The computation of the explicit value of K_n is classical (see for instance [46]).

Thus, in order to get the estimate of the theorem and using the min-max principle (1.1), we look for $r \in (-1, 1)$ such that for all $(x, y) \in \mathbf{R}^2 \setminus \{0\}$:

$$q(x, y) < 2^{\frac{2}{n}}.$$

Since $I > 1$ and $J_r > 1$, we look for $r \in (-1, 1)$ such that

$$(\sigma - 2^{\frac{2}{n}})x^2 + 2(\alpha_r - 2^{\frac{2}{n}}\beta_r)yx + (\tau_r - 2^{\frac{2}{n}})y^2 < 0.$$

Moreover, since $\sigma < 1$ and $\tau_r - 2^{\frac{2}{n}} < 0$, it is sufficient to find $r \in (-1, 1)$ such that

$$\alpha_r - 2^{\frac{2}{n}}\beta_r = 0.$$

By the claim 4, we know that

$$\alpha_r = \frac{-\int_{\mathbf{S}^n} \psi_r (\Delta_g \phi) dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}} \xrightarrow{r \rightarrow 1} \frac{-\int_{\mathbf{S}^n} \phi (\Delta_g \phi) dv_g}{\left(\int_{\mathbf{S}^n} |\nabla_g \phi|_g^n dv_g \right)^{\frac{2}{n}}} = \sigma$$

and that

$$\beta_r = (n+1) \int_{\mathbf{S}^n} \phi \psi_r dv_g \xrightarrow{r \rightarrow 1} (n+1) \int_{\mathbf{S}^n} \phi^2 dv_g = I.$$

Thus, when $r \rightarrow 1$ and in an analogous way, when $r \rightarrow -1$, (see claim 4),

$$\alpha_r - 2^{\frac{2}{n}}\beta_r \xrightarrow{r \rightarrow 1} \sigma - 2^{\frac{2}{n}}I < 0$$

and

$$\alpha_r - 2^{\frac{2}{n}}\beta_r \xrightarrow{r \rightarrow -1} 2^{\frac{2}{n}}I - \sigma > 0.$$

By continuity, (claim 4), there exists $r \in (-1, 1)$ such that $\alpha_r - 2^{\frac{2}{n}}\beta_r = 0$. As already said, this completes the proof of the theorem.

Chapitre 2

Résultat de rigidité sur la première valeur propre conforme

Etant donnée une variété Riemannienne compacte sans bord (M, g) de dimension $n \geq 3$, on considère la première valeur propre conforme qui est par définition la borne supérieure de la première valeur propre du Laplacien parmi toutes les métriques conformes à g de volume 1. Nous démontrons dans ce chapitre qu'elle est toujours plus grande que $n\omega_n^{\frac{2}{n}}$, la valeur qu'elle prend dans la classe conforme de la sphère ronde, sauf si (M, g) est conforme à la sphère standard.

Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 3$ and let us define the first conformal eigenvalue of (M, g) by

$$\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(M, \tilde{g}) \text{Vol}_{\tilde{g}}(M)^{\frac{2}{n}}$$

where $\lambda_1(M, g)$ is the first nonzero eigenvalue of the Laplacian $\Delta_g = -\text{div}_g(\nabla)$ and $[g]$ is the conformal class of g . In this paper, we aim at proving a rigidity result concerning this first conformal eigenvalue.

The maximisation on conformal classes is natural because the scale invariant quantity supremum is infinite among all metrics [20] (except in dimension 2, [113]), while El Soufi and Ilias [33] proved that it is always bounded among conformal metrics. Generalizing a result by Li and Yau [74] in dimension 2, they gave an explicit upper bound thanks to the m -conformal volume $V_c(m, M, [g])$ of $(M, [g])$

$$\Lambda_1(M, [g]) \leq n V_c(m, M, [g])^{\frac{2}{n}} \quad (2.1)$$

These conformal invariants on the standard sphere $(\mathbb{S}^n, [can])$ satisfy, [33]

$$\Lambda_1(\mathbb{S}^n, [can]) = n \omega_n^{\frac{2}{n}} = n V_c(\mathbb{S}^n, [can])^{\frac{2}{n}} \quad (2.2)$$

and this value is achieved if and only if the metric is round. Here, ω_n denotes the volume of the standard n -sphere. Colbois and El Soufi [21] also proved that, for any compact Riemannian manifold (M, g) of dimension $n \geq 3$

$$\Lambda_1(M, [g]) \geq \Lambda_1(\mathbb{S}^n, [can]) .$$

We prove here that the case of equality characterizes the standard sphere :

Theorem 2. *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$. Then*

$$\Lambda_1(M, [g]) > \Lambda_1(\mathbb{S}^n, [can])$$

if $(M, [g])$ is not conformally diffeomorphic to $(\mathbb{S}^n, [can])$.

This theorem answers the question raised in [19] and [66]. Note that a similar result was proved by the author in dimension 2 (see [91]). Note also that thanks to (2.1) and (2.2), the theorem implies

$$V_c(m, M, [g]) > \omega_n = V_c(\mathbb{S}^n, [can])$$

if $(M, [g])$ is not conformally diffeomorphic to $(\mathbb{S}^n, [can])$. This gives a positive answer to question 2 in [74].

In the rest of this paper, we prove the theorem. Based on the idea of Ledoux [73] and Druet [27], we start from a sharp Sobolev inequality in dimensions $n \geq 3$ (see [50, 27, 29]) which possesses extremal functions. These extremal functions give natural metrics $\tilde{g} \in [g]$ with $\text{Vol}_{\tilde{g}}(M) = 1$ and $\lambda_1(\tilde{g}) \geq n \omega_n^{\frac{2}{n}}$. As in dimension 2, see [91], we deal with the degeneracy consequences of the hypothesis $\lambda_1(\tilde{g}) = n \omega_n^{\frac{2}{n}}$.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with $\text{Vol}_g(M) = 1$, which is not conformally diffeomorphic to the standard sphere. For an integer $m \geq 1$, let $h \in \mathcal{C}^m(M)$. We let $J_{g,h}$ be the functional defined for $u \in W^{1,2}(M) \setminus \{0\}$ by

$$J_{g,h}(u) = \frac{\int_M |\nabla u|_g^2 dv_g + \int_M hu^2 dv_g - K_n^{-2} \left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}}{\int_M u^2 dv_g} \quad (2.3)$$

where

$$K_n = \frac{2}{\sqrt{n(n-2)}} \omega_n^{-\frac{1}{n}} \quad (2.4)$$

is the sharp constant for the Sobolev inequality induced by the critical Sobolev embedding $W_0^{1,2} \subset L^{2^*}$ for bounded domains of \mathbb{R}^n , with $2^* = \frac{2n}{n-2}$. Hebey and Vaugon proved in [50] that

$$-\alpha(g, h) = \inf_{u \in W^{1,2}(M) \setminus \{0\}} J_{g,h}(u) \quad (2.5)$$

is finite. Note that $J_{g,h}$ is scale invariant.

We will assume in the following that up to a conformal change, g is a metric in $[g]$ with volume 1 which has a constant scalar curvature S_g . Since M is not conformally diffeomorphic to the standard sphere, by the resolution of the Yamabe problem by Aubin [3] and Schoen [103], it satisfies

$$\mu(M, g) < K_n^{-2} \quad (2.6)$$

where $\mu(M, g)$ is the Yamabe invariant of $(M, [g])$. Let V be an open neighbourhood of $\frac{n-2}{4(n-1)} S_g$ in $\mathcal{C}^m(M)$ such that

$$\forall h \in V, \left\| h - \frac{n-2}{4(n-1)} S_g \right\|_\infty \leq \frac{1}{2} (K_n^{-2} - \mu(M, g)) . \quad (2.7)$$

Let $s \geq 0$ be such that $s+2 > \frac{n}{2}$ and $m \geq s+2$. By the Sobolev embedding $W^{s+2,2} \hookrightarrow \mathcal{C}^0$, the subset $W_+^{s+2,2}$ of positive functions of $W^{s+2,2}$ is open. We define

$$\begin{aligned} F : W_+^{s+2,2} \times \mathbb{R} \times V &\longrightarrow W^{s,2} \\ (u, \beta, h) &\longmapsto \Delta_g u + (h + \beta)u - K_n^{-2} u^{2^*-1} \end{aligned}$$

which is well defined because of the Sobolev algebra property of $W^{s+2,2}$ and F is a \mathcal{C}^∞ map. By a result of Druet [27], thanks to (2.6) and (2.7), for any $h \in V$, the functional $J_{g,h}$ attains its infimum. Let $u \in W^{1,2}(M)$ be such that $J_{g,h}(u) = -\alpha(g, h)$. Up to replace u by $|u|$ and up to normalize, we can take $u \geq 0$ and $\int_M u^{2^*} dv_g = 1$. Then, u satisfies the Euler-Lagrange equation

$$F(u, \alpha(g, h), h) = \Delta_g u + (h + \alpha(g, h))u - K_n^{-2} u^{2^*-1} = 0 \quad (2.8)$$

where, by elliptic regularity theory, $u \in \mathcal{C}^{m+2}$ and, by the maximum principle, $u > 0$.

Let $v \in \mathcal{C}^\infty(M)$ and $t \in \mathbb{R}$ such that $|t| < \|v\|_\infty^{-1}$. Since u is a minimum for (2.5),

$$\begin{aligned} \int_M |\nabla(u + tuv)|_g^2 dv_g + \int_M (h + \alpha(g, h))(u + tuv)^2 dv_g \\ - K_n^{-2} \left(\int_M (u + tuv)^{2^*} dv_g \right)^{\frac{2}{2^*}} \geq 0 . \end{aligned} \quad (2.9)$$

Since u satisfies (2.8), the left term in (2.9) vanishes until the order 2 in the Taylor development as $t \rightarrow 0$. Computing the second-order coefficient as $t \rightarrow 0$, one gets

$$\begin{aligned} \int_M |\nabla(uv)|_{\tilde{g}}^2 dv_g + \int_M (h + \alpha(g, h))(uv)^2 dv_g - K_n^{-2}(2^* - 1) \int_M v^2 u^{2^*} dv_g \\ + K_n^{-2}(2^* - 2) \left(\int_M v u^{2^*} dv_g \right)^2 \geq 0. \end{aligned} \quad (2.10)$$

We now use the conformal transformation of the conformal Laplacian

$$\forall v \in C^\infty(M), u^{2^*-1} \Delta_{\tilde{g}} v = \Delta_g(uv) - v \Delta_g u \quad (2.11)$$

where $\tilde{g} = u^{\frac{4}{n-2}} g$. We integrate (2.11) against uv and with (2.8),

$$\begin{aligned} \int_M |\nabla(uv)|_{\tilde{g}}^2 dv_g &= \int_M |\nabla v|_{\tilde{g}}^2 dv_{\tilde{g}} + \int_M v^2 u \Delta_g u dv_g \\ &= \int_M |\nabla v|_{\tilde{g}}^2 dv_{\tilde{g}} - \int_M (h + \alpha(g, h)) v^2 u^2 dv_g + K_n^{-2} \int_M v^2 u^{2^*} dv_g \end{aligned}$$

and with (2.4), (2.10) becomes

$$\int_M |\nabla v|_{\tilde{g}}^2 dv_{\tilde{g}} - n \omega_n^{\frac{2}{n}} \int_M \left(v - \int_M v dv_{\tilde{g}} \right)^2 dv_{\tilde{g}} \geq 0. \quad (2.12)$$

This gives that $\lambda_1(\tilde{g}) \geq n \omega_n^{\frac{2}{n}}$. Note that if the inequality is strict for one solution (h, u) of $F(u, \alpha(g, h), h) = 0$, the theorem is proved.

We now assume that for any solution (h, u) of $F(u, \alpha(g, h), h) = 0$, we have $\lambda_1(u^{\frac{4}{n-2}} g) = n \omega_n^{\frac{2}{n}}$. We will apply the following theorem ([53], Theorem 5.4, page 63) of Fredholm theory to F , with $U = W_+^{s+2,2}(M) \times \mathbb{R}$.

Theorem 3. Let X, Y be two separable Banach spaces, U an open set of X , V a separable C^∞ Banach manifold and $F \in C^\infty(U \times V, Y)$ which satisfy :

- For all $(u, v) \in F^{-1}(0)$, $DF(u)$ is surjective.
- For all $(u, v) \in F^{-1}(0)$, $D_u F(u, v)$ is a Fredholm operator.

Then there exists a countable intersection of open dense sets (a residual set) $\Sigma \subset V$ such that for all $v \in \Sigma$, and for all $u \in F(., v)^{-1}(0)$, $D_u F(u, v)$ is surjective.

Using (2.11) and (2.4), one gets for $(u, \beta, h) \in F^{-1}(0)$,

$$D_{(u, \beta)} F(u, \beta, h).(\theta, \mu) = u^{2^*-1} \left(\Delta_{\tilde{g}} \left(\frac{\theta}{u} \right) - n \omega_n^{\frac{2}{n}} \frac{\theta}{u} \right) + \mu u \quad (2.13)$$

where $\tilde{g} = u^{\frac{4}{n-2}} g$. Then, $D_{(u, \beta)} F(u, \beta, h)$ is a Fredholm operator. It remains to prove that if $(u, \beta, h) \in F^{-1}(0)$, $DF(u, \beta, h)$ is surjective. We have

$$DF(u, \beta, h).(\theta, \mu, \tau) = u^{2^*-1} \left(\Delta_{\tilde{g}} \left(\frac{\theta}{u} \right) - n \omega_n^{\frac{2}{n}} \frac{\theta}{u} \right) + \mu u + \tau u. \quad (2.14)$$

$Im(D_{(u, \beta)} F(u, \beta, h))$ is a closed space in $W^{s,2}$ of finite codimension. Thus, since $Im(DF(u, \beta, h))$ contains $Im(D_{(u, \beta)} F(u, \beta, h))$, it is a closed space in $W^{s,2}$ by the following

Lemma. *Let X a banach space, and $E \subset F \subset X$ some subspaces. If E is a closed finite co-dimentional subbase of X , then F is a closed subspace of X .*

Proof. Let G a finite dimensional subspace of X such that $X = E \oplus G$. We set $H = G \cap F$. Then, $F = E \oplus H$. Let $x_k \in F$ such that $x_k \rightarrow x$ as $k \rightarrow +\infty$. We denote $x_k = y_k + z_k$ with $y_k \in E$ and $z_k \in H$.

We suppose that $(z_k)_{k \geq 0}$ is not bounded. Then, up to the extraction of a subsequence, $|z_k| \rightarrow +\infty$ as $k \rightarrow +\infty$. By Bolzano's theorem, up to the extraction of a subsequence, there exists $z \in H$ such that

$$\frac{z_k}{|z_k|} \rightarrow z \text{ as } k \rightarrow +\infty .$$

Since (x_k) converges as $k \rightarrow +\infty$,

$$\frac{y_k}{|z_k|} = \frac{x_k}{|z_k|} - \frac{z_k}{|z_k|} \rightarrow -z \text{ as } k \rightarrow +\infty .$$

Since E is closed, we get $z \in E \cap H = 0$, which contradicts $|z| = 1$.

Then $(z_k)_{k \geq 0}$ is bounded and by Bolzano's theorem, up to the extraction of a subsequence, we can suppose that $z_k \rightarrow z \in H$ as $k \rightarrow +\infty$. Then,

$$y_k = x_k - z_k \rightarrow x - z \text{ as } k \rightarrow +\infty .$$

and $y = x - z \in E$ since E is closed. Therefore $x = y + z \in E + H = F$ and the proof of the lemma is complete. \diamond

Now, it suffices to prove that $\text{Im}(DF(u, \beta, h))^\perp = 0$, where \perp refers to the orthogonal in $W^{s,2}$. Let $\phi \in \text{Im}(DF(u, \beta, h))^\perp$. Then, with (2.14),

$$\forall \tau \in \mathcal{C}^m, \langle \phi, u\tau \rangle_{W^{s,2}} = 0 .$$

Since $u \in \mathcal{C}^m$ is positive and \mathcal{C}^m is dense in $W^{s,2}$, we get $\phi = 0$.

By Theorem 3, there exists $h \in V$ such that for all couple (u, β) with $F(u, \beta, h) = 0$, $DF_{(u,\beta)}(u, \beta, h)$ is surjective. We take in particular $\beta = \alpha(g, h)$ and we will deduce that for a minimal function u , $\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}$ is simple with $\tilde{g} = u^{\frac{4}{n-2}}g$. We claim that

$$\forall \phi \in E_1(\tilde{g}) \setminus \{0\}, \int_M u^2 \phi dv_g \neq 0 . \quad (2.15)$$

Indeed, if ϕ is an eigenfunction for $\lambda_1(\tilde{g})$ such that this integral vanishes, one easily checks with (2.13) that $u\phi$ is orthogonal to the image of $D_{(u,\beta)}F(u, \alpha(h, g), h)$ in $L^2(g)$. It implies $\phi = 0$ and we obtain (2.15). Since a bounded linear form vanishes on a one-codimensional space, we get that $\lambda_1(\tilde{g})$ is simple. Thus, $\lambda_1(\tilde{g})$ cannot be an extremal eigenvalue in the sense of [35] and as a result, $\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}$ is not locally maximal. The proof of Theorem 2 for $n \geq 3$ is complete.

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Chapitre 3

Existence et régularité de métriques maximales pour la première valeur propre du laplacien sur des surfaces

Nous cherchons à démontrer dans ce chapitre l'existence de métriques qui maximisent la première valeur propre du Laplacien sur des surfaces riemanniennes. Nous prouvons d'abord le résultat de rigidité énoncé dans le chapitre 2 dans le cas de la dimension 2. Ceci permet de démontrer que, dans une classe conforme donnée, il existe toujours une métrique maximale pour la première valeur propre de Laplace qui est de classe \mathcal{C}^∞ sauf en un nombre fini de points de singularités coniques. Ensuite, nous démontrons des résultats d'existence parmi toutes les métriques sur des surfaces de genre donné, menant à l'existence d'immersions minimales de variétés compactes (M, g) de dimension 2 dans une certaine k -sphère par des premières fonctions propres. Ce résultat est similaire à celui concernant les valeurs propres de Steklov obtenu par Fraser et Schoen [41]. Enfin, nous répondons à une conjecture de Friedlander et Nadirashvili [42], qui stipule qu'on peut se donner dans certaines classes conformes la borne supérieure de la première valeur propre du Laplacien aussi proche qu'on veut de sa valeur sur une sphère, sur toute surface orientable.

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3.1 Introduction

Let (Σ, g) be a smooth compact Riemannian surface without boundary. The eigenvalues of the Laplacian $\Delta_g = -\operatorname{div}_g(\nabla)$ form a discrete sequence

$$0 = \lambda_0 < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \dots$$

Getting bounds on these eigenvalues depending on the metric or the topology of Σ has been the subject of intensive studies in the past decades. In this paper, we shall focus on the first eigenvalue λ_1 . One can for instance consider the first conformal eigenvalue of (Σ, g) defined by

$$\Lambda_1(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \operatorname{Vol}_{\tilde{g}}(\Sigma) . \quad (3.1)$$

If one looks at the infimum of the first eigenvalue in a given conformal class, it is always 0. Now one can also study invariants which depend only on the topology of the surface. For orientable surfaces, one can define for any genus $\gamma \geq 0$

$$\Lambda_1(\gamma) = \sup_g \lambda_1(g) \operatorname{Vol}_g(\Sigma) = \sup_{[g]} \Lambda_1(\Sigma, [g]) \quad (3.2)$$

where Σ is a compact orientable surface of genus γ . One can also look at

$$\inf_{[g]} \Lambda_1(\Sigma, [g]) .$$

Natural questions about these quantities are to get explicit values or explicit bounds on it, and whether or not the supremum (or infimum) in their definition is achieved by some metric and, if yes, how regular these extremal metrics are. Yang and Yau [113] (see also [74]) obtained an upper-bound for the first eigenvalue of the Laplacian on a surface, depending only on the genus γ of the surface. In case of orientable surfaces, this reads as

$$\Lambda_1(\gamma) \leq 8\pi \left\lceil \frac{\gamma + 3}{2} \right\rceil . \quad (3.3)$$

Colbois and El Soufi [21] gave an explicit lower bound of $\Lambda_1(\Sigma, [g])$ on any closed Riemannian surface and proved that

$$\Lambda_1(\Sigma, [g]) \geq \Lambda_1(\mathbb{S}^2, [\text{can}])$$

and by the work of Hersch [54], we know that $\Lambda_1(\mathbb{S}^2, [\text{can}]) = 8\pi$. A lower bound for $\Lambda_1(\gamma)$ can be obtained from [10] and [14] (see [39]) :

$$\Lambda_1(\gamma) \geq \frac{3\pi}{4}(\gamma - 1) . \quad (3.4)$$

Exact values of these quantities were obtained for small genus and for specific conformal classes. Let us mention the sphere (Hersch [54]), the projective plane (Li-Yau [74]), the torus (Girouard [45] and Nadirashvili [83]), the Klein bottle (El Soufi-Giacomini-Jazar [32] and Jakobson-Nadirashvili-Polterovitch [57]), the genus 2 surfaces (Jakobson-Levitin-Nadirashvili-Nigam-Polterovitch [56]).

Concerning $\Lambda_1(\Sigma, [g])$, we prove the following theorem :

Theorem 4. *Let (Σ, g) be a compact Riemannian surface without boundary. Then*

$$\Lambda_1(\Sigma, [g]) > \Lambda_1(\mathbb{S}^2, [\text{can}]) = 8\pi$$

if Σ is not diffeomorphic to \mathbb{S}^2 . Moreover, there is an extremal metric $\tilde{g} \in [g]$, smooth except maybe at a finite number of points corresponding to conical singularities, such that $\Lambda_1(\Sigma, [g]) = \lambda_1(\tilde{g}) \text{Vol}_{\tilde{g}}(\Sigma)$.

This theorem contains a rigidity result which states that the sphere is characterized by having the minimal first conformal eigenvalue. It also contains an existence result of "smooth" maximal metrics. Note that, on the sphere, we know since the work of Hersch [54] that maximal metrics exist and are all smooth since they consist in all metrics isometric to the standard one. As observed in [65], conical singularities naturally appear for extremal metrics. Indeed, the conformal factor relating \tilde{g} to g is $|\nabla\Phi|_g^2$ where Φ is some smooth harmonic map from M into some sphere \mathbb{S}^k . The zeros of $|\nabla\Phi|_g^2$ are isolated and correspond to branch points of the harmonic map Φ as proved in Salamon [101]. In the case of genus 2 surface (see [56]), the extremal metrics of the conjecture indeed possess conical singularities. In this respect, our existence result seems completely optimal. In [65], Kokarev proved that any maximizing sequence of metrics, provided that our rigidity result was true, converges to a Radon measure without atoms. He then got some partial regularity results on this measure. Note that, here, we do not prove that any maximizing sequence converges to a "smooth" maximizer, which may not be true. We select carefully a maximizing sequence which converges to a "smooth" maximizer. In [85], assuming that $\Lambda_1(\Sigma, [g]) > 8\pi$, which is by now a consequence of our result, the authors announced the existence of a maximizer with a rather different proof we do not fully understand.

Note also that, by Kokarev [64], and thanks to the rigidity part of our theorem, we know that the set of "smooth" maximizers given by our theorem is compact as soon as M is not diffeomorphic to the sphere \mathbb{S}^2 . On the sphere \mathbb{S}^2 , this compactness result is of course false.

Capitalizing on this first existence result, we are also able to obtain the following :

Theorem 5. *Let Σ be a compact orientable Riemannian surface without boundary of genus $\gamma \geq 1$. If $\Lambda_1(\gamma) > \Lambda_1(\gamma - 1)$, then $\Lambda_1(\gamma)$ is achieved by a metric which is smooth except at a finite set of conical singularities.*

Note that the case of the sphere is already treated in Hersch [54]. Note also that the case of the torus ($\gamma = 1$) is already known : we have $\Lambda_1(1) = \frac{8\pi^2}{\sqrt{3}}$ and the maximal metric is given by the flat equilateral torus (see [83]). At last, in the genus 2 case, a conjecture holds : $\Lambda_1(2) = 16\pi$ and there is a family of maximal metrics (see [56]).

The spectral gap $\Lambda_1(\gamma) > \Lambda_1(\gamma - 1)$ necessarily holds for an infinite number of γ thanks to the lower bound (3.4). It is believed to hold for all genres. The extremal metric in the theorem is the pull-back of the induced metric of a minimal immersion (with branched points) of Σ into some sphere S^k . As a classical corollary of the above theorem, we obtain the following :

Corollary. *If $\gamma \geq 1$ and if $\Lambda_1(\gamma) > \Lambda_1(\gamma - 1)$, which is the case at least for an infinite number of γ , there exists a minimal immersion (possibly with branch points) of a compact surface Σ of genus γ into some sphere S^k by first eigenfunctions.*

There have been lot of works about minimal immersions of surfaces into spheres. In particular, they are necessarily given by eigenfunctions (not only first eigenfunctions) thanks to Takahashi [108]. For existence results of such immersions, we refer to two classical papers by Lawson [72] and Bryant [11]. Concerning minimal embeddings in S^3 , it is conjectured by Yau [114] that they all come from first eigenfunctions (see [7] and [17] for recent surveys on this subject). However, minimal immersions by first eigenfunctions are not so numerous. For instance, it has been proved by Montiel and Ros [80] that there is at most one minimal immersion by first eigenfunctions in any given conformal class. In the case of genus 1, it was also proved by El Soufi and Ilias [34] that the only minimal immersions by first eigenfunctions of the torus are the Clifford torus (in S^3) and the flat equilateral torus (in S^5). So our corollary is interesting because it provides an infinite number of new minimal immersions into spheres by first eigenfunctions.

At last, we prove a conjecture stated in [42] about the infimum of the first conformal eigenvalue on any orientable surface :

Theorem 6. *Let Σ be a smooth compact orientable surface. Then*

$$\inf_{[g]} \Lambda_1(\Sigma, [g]) = 8\pi$$

and this infimum is never attained except on the sphere.

This result had already been proved in [45] in genus 1 but was left open in higher genres up to now.

The paper is organized as follows :

We first prove in section 3.2 the rigidity part of theorem 4. The idea of the proof goes back to Ledoux [73] and Druet [27] in higher dimensions. We start from some Moser-Trudinger type inequality (see [16, 26, 81]) which possesses extremal functions. These extremal functions are excellent candidates to provide conformal factors for which the new metric has a large λ_1 . However, we have to deal with some degeneracy problems which could occur.

Then, we prove the existence of a "smooth" extremal metric for $\Lambda_1(M, [g])$. In section 3.3.1, we prove some fine non-concentration estimates for sequences of unit volume metrics in a given conformal class with large first eigenvalue. This non-concentration phenomenon was

first observed by Girouard [45] and Kokarev [65]. Section 3.3.2 is devoted to the construction of our specific maximizing sequence, following ideas of Fraser and Schoen [41] when dealing with the Steklov eigenvalue problem. It is obtained by solving a regularized maximization problem. We derive a fine Euler-Lagrange characterization for this new variational problem. This leads to a maximizing sequence of smooth metrics for which the first eigenspace possesses nice properties. Section 3.3.3 makes an intensive use of the non-concentration estimates of section 3.3.1 to get finer and finer estimates on these first eigenfunctions. This permits then to pass to the limit and to prove theorem 4.

Section 3.4 is devoted to the proof of theorem 5. Since we already have the existence of a maximizing metric in any given conformal class thanks to theorem 4, it remains to prove that the supremum among all conformal classes is achieved. For that purpose, we pick up a sequence of maximizing conformal classes and prove that this sequence does not degenerate. We follow ideas of Zhu [115] who made a careful study of sequence of harmonic maps into spheres on hyperbolic surfaces which degenerate.

The last section is devoted to the proof of theorem 6. It is in some sense similar to the proof of theorem 5, except that we just have to construct a sequence of fully degenerating conformal classes (c_α) on some hyperbolic surface Σ for which we prove that $\Lambda_1(\Sigma, c_\alpha) \rightarrow 8\pi$.

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3.2 The rigidity result

3.2.1 Extremal functions for a sharp Moser-Trudinger inequality

Let (M, g) be a smooth compact Riemannian surface with $\text{Vol}_g(M) = 1$, not diffeomorphic to \mathbb{S}^2 . The aim of this section is to find a metric $\tilde{g} \in [g]$ such that $\lambda_1(\tilde{g}) > 8\pi$. We consider the functional J_g defined for $u \in H_1^2(M)$ by

$$J_g(u) = \frac{1}{4\pi} \int_M |\nabla u|_g^2 dv_g + 2 \int_M u dv_g - \log \left(\int_M e^{2u} dv_g \right). \quad (3.5)$$

Cherrier [16] proved that

$$-\alpha(g) = \inf_{u \in H_1^2(M)} J_g(u) \quad (3.6)$$

is always finite. Note that J_g is translation invariant with respect to the constant functions.

In the following, up to a harmless conformal change of the metric, we assume that g is the metric in $[g]$ with volume 1 and constant Gaussian curvature $K_g \equiv K_0$. Since M is not diffeomorphic to the sphere, we know that $K_0 \leq 2\pi$. Then there exists a ball $B_\delta(0) \subset C^2(M)$ centered at 0 and of radius $\delta > 0$ such that for any $v \in B_\delta(0)$,

$$K_{e^{2v}g} < 4\pi.$$

We claim that there exists $v_0 \in B_\delta(0) \subset C^2(M)$ such that 8π is not in the spectrum of $\Delta_{e^{2v_0}g}$. It is completely obvious since the spectrum is discrete and it scales like the volume to the power -1 . Thus there exists an open set $V \subset C^2(M)$ such that, for any $v \in V$,

$$K_{e^{2v}g} < 4\pi \quad (3.7)$$

and

$$8\pi \text{ is not in the spectrum of } \Delta_{e^{2v}g}. \quad (3.8)$$

Let us now define the following map :

$$\begin{aligned} F : W^{2,2}(M) \times V &\longrightarrow L^2(M) \\ (u, v) &\longmapsto e^{-2v}\Delta_g u + 4\pi - 4\pi e^{2u} \end{aligned}$$

It is well defined because of the Sobolev embedding $W^{2,2}(M) \subset C^0(M)$ (see [43], Corollary 7.11, page 158) and F is a C^∞ map.

By a result of [26] and thanks to (3.7), for any $v \in V$, the functional $J_{e^{2v}g}$ attains its minimum. Let $u \in H_1^2(M)$ be such that $J_{e^{2v}g}(u) = -\alpha(e^{2v}g)$ normalized by $\int_M e^{2u} e^{2v} dv_g = 1$. Then $u \in C^3(M)$ and satisfies the Euler-Lagrange equation

$$F(u, v) = e^{-2v}\Delta_g u + 4\pi - 4\pi e^{2u} = 0. \quad (3.9)$$

Moreover, we have that

$$D^2 J_{e^{2v}g}(u)(\varphi, \varphi) = \frac{1}{2\pi} \left(\int_M |\nabla \varphi|_{\tilde{g}}^2 dv_{\tilde{g}} - 8\pi \int_M \varphi^2 dv_{\tilde{g}} + 8\pi \left(\int_M \varphi dv_{\tilde{g}} \right)^2 \right) \geq 0$$

where $\tilde{g} = e^{2(u+v)}g$. This means in particular that $\lambda_1(e^{2(u+v)}g) \geq 8\pi$. Since

$$D_u F(u, v)(\varphi) = e^{2u} (\Delta_{e^{2(u+v)}g} \varphi - 8\pi \varphi), \quad (3.10)$$

we have $\lambda_1(e^{2(u+v)}g) > 8\pi$ as soon as $D_u F(u, v)(\varphi)$ is invertible. Thus, in order to prove the rigidity part of the theorem, we just need to find $v \in V$ such that $D_u F(u, v)$ is invertible for all solutions u of $F(u, v) = 0$.

For that purpose, we shall apply the following theorem of Fredholm theory (see for instance [53], Theorem 5.4, page 63) to our function F :

Theorem 7. *Let X, Y two separable Banach spaces, U an open set of X , V a separable C^∞ Banach manifold and $F \in C^\infty(U \times V, Y)$ which satisfies :*

- *For all $(u, v) \in F^{-1}(0)$, $DF(u, v)$ is surjective.*
- *For all $(u, v) \in F^{-1}(0)$, $D_u F(u, v)$ is a Fredholm operator.*

Then, there exists a countable intersection of open dense sets (a residual set) $\Sigma \subset V$ such that for all $v \in \Sigma$, and for all $u \in F(., v)^{-1}(0)$, $D_u F(u, v)$ is surjective.

By (3.10), it is clear that $D_u F(u, v)$ is a Fredholm operator. It remains to prove that if $(u, v) \in F^{-1}(0)$, $DF(u, v)$ is surjective. We have

$$DF(u, v).(\theta, \tau) = e^{2u} (\Delta_{\tilde{g}} \theta - 8\pi \theta) - 2\tau e^{-2v} \Delta_g u$$

where $\tilde{g} = e^{2(u+v)}g$. Since the image of $DF(u, v)$ contains a finite codimensional closed space, it is a closed space. Moreover, the L^2 norms induced by the metrics g and $e^{2v}g$ are equivalent. Then, it suffices to prove that the orthogonal of the image is 0 in $L^2(e^{2v}g)$. Assume on the contrary that there exists $\phi \in L^2(M)$, $\phi \not\equiv 0$, such that

$$\forall \theta \in W^{2,2}(M), \int_M (\Delta_{\tilde{g}} \theta - 8\pi \theta) \phi dv_{\tilde{g}} = 0$$

and

$$\forall \tau \in C^2(M), \int_M \phi \tau \Delta_g u dv_g = 0.$$

The first condition implies that ϕ is an eigenfunction for \tilde{g} with eigenvalue 8π . We deduce from the second condition knowing that $e^{-2v}\Delta_g u + 4\pi = 4\pi e^{2u}$ that

$$\phi(e^{2u} - 1) = 0.$$

Since ϕ is non-zero on a dense set of points by the maximum principle, $e^{2u} \equiv 1$. This implies that $\tilde{g} = e^{2v}g$ has a 8π eigenvalue, which contradicts (3.8).

Now, we can apply the above theorem to our function F . There exists some $v \in V$ such that $D_u F(u, v)$ is surjective for all $u \in W^{2,2}(M)$ such that $F(u, v) = 0$. Since $D_u F(u, v)$ is a Fredholm operator of index 0, see (3.10), it is also injective. As already said, this ends the proof of the rigidity part of theorem 4. For such a v and for a minimal function u for $J_{e^{2v}g}$, we get that $\lambda_1(e^{2(u+v)}g) > 8\pi$.

3.3 Existence of maximal metrics in a conformal class

Let (M, g) be a smooth compact unit volume Riemannian surface without boundary. We choose for all the proof some $\delta > 0$, some $C_0 > 1$, a family $(x_i)_{i=1, \dots, N}$ of points in M and smooth functions $v_i : M \mapsto \mathbb{R}$ such that

- for any $i \in \{1, \dots, N\}$, the Gauss curvature of $g_i = e^{2v_i}g$ is 0 in the ball $B_{g_i}(x_i, 2\delta) = \Omega_i$ so that, in the exponential chart for the metric g_i at x_i , the metric g_i is the Euclidean metric.
- $M = \bigcup_{i=1}^N \omega_i$ where $\omega_i = B_{g_i}(x_i, \delta)$.
- for any $i \in \{1, \dots, N\}$, $C_0^{-2} \leq e^{2v_i} \leq C_0^2$ in Ω_i .

Note that, for any $i \in \{1, \dots, N\}$,

$$B_g(x, C_0^{-1}r) \subset B_{g_i}(x, r) \subset B_g(x, C_0 r) \text{ for all } x \in \omega_i \text{ and all } 0 < r \leq \delta. \quad (3.11)$$

During all this section, in order to get uniform estimates, we may assume, without loss of generality that every sequence $\{x_\epsilon\}$ of points of M lies in some ω_i where i is fixed. Indeed, every subsequence of $\{x_\epsilon\}$ has a subsequence which satisfies this property.

3.3.1 Non-concentration estimates of metrics with high first eigenvalue

In this subsection, we let $\{e^{2u_\epsilon}g\}_\epsilon$ be a sequence of unit volume metrics such that

$$\lambda_1(e^{2u_\epsilon}g) \geq 8\pi + \alpha \text{ for some } \alpha > 0 \text{ fixed.} \quad (3.12)$$

Note that we also know that $\lambda_1(e^{2u_\epsilon}g) \leq \Lambda_1(M, [g])$. By Kokarev [65], lemma 2.1 and lemma 3.1, the following non-concentration result follows from (3.12) :

Claim 5 (Kokarev [65]). *Assume that (u_ϵ) is a sequence of smooth functions on M such that (3.12) holds. Then*

$$\lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \sup_{x \in M} \int_{B_g(x,r)} e^{2u_\epsilon} dv_g = 0.$$

We set for Ω an open subset of M and $\mu \in \mathcal{M}(M)$, the set of Radon measures on M ,

$$\lambda_\star(\Omega, \mu) = \inf_{\phi \in \mathcal{C}_c^\infty(\Omega)} \frac{\int_\Omega |\nabla \phi|_g^2 dv_g}{\int_\Omega \phi^2 d\mu}$$

and

$$\beta(\Omega, \mu) = \sup \left\{ \frac{\mu(K)}{\text{Cap}_2(K, \Omega)}; K \subset \Omega \text{ is a compact set} \right\}$$

where, for a compact set $K \subset \Omega$, $\text{Cap}_2(K, \Omega)$ is the variational capacity of (K, Ω) defined by

$$\text{Cap}_2(K, \Omega) = \inf \left\{ \int_\Omega |\nabla \phi|_g^2 dv_g; \phi \in \mathcal{C}_c^\infty(\Omega), \phi = 1 \text{ on } K \right\}.$$

Isocapacity inequalities proved in [77], section 2.3.3, corollary of theorem 2.3.2, give that

$$\frac{1}{4\beta(\Omega, \mu)} \leq \lambda_\star(\Omega, \mu) \leq \frac{1}{\beta(\Omega, \mu)}. \quad (3.13)$$

Thanks to these capacity estimates, we can refine the non-concentration result of Kokarev and obtain a quantitative one :

Claim 6. *Assume that (u_ϵ) is a sequence of smooth functions on M such that (3.12) holds. Then there exists $C_1 > 0$ such that*

$$\int_{B_g(x,r)} e^{2u_\epsilon} dv_g \leq \frac{C_1}{\ln \frac{1}{r}}$$

for all $\epsilon > 0$ and all $r > 0$.

Proof. We first prove that there exists $r_0 > 0$ such that for any $0 < r \leq r_0$,

$$\forall \epsilon > 0, \forall x \in M, \frac{1}{\lambda_\star(B_g(x, r), e^{2u_\epsilon}g)} \leq \frac{2}{\lambda_1(M, e^{2u_\epsilon}g)} \leq \frac{1}{4\pi}. \quad (3.14)$$

Indeed, choose $\psi_\epsilon \in E_1(B_g(x, r_0), e^{2u_\epsilon}g)$ with $\int_M \psi_\epsilon^2 e^{2u_\epsilon} dv_g = 1$ and let us write that

$$\begin{aligned} & \int_{B_g(x, r_0)} \psi_\epsilon^2 e^{2u_\epsilon} dv_g - \left(\int_{B_g(x, r_0)} \psi_\epsilon e^{2u_\epsilon} dv_g \right)^2 \\ & \leq \frac{1}{\lambda_1(M, e^{2u_\epsilon}g)} \int_M |\nabla \psi_\epsilon|_g^2 dv_g = \frac{\lambda_\star(B_g(x, r_0), e^{2u_\epsilon}g)}{\lambda_1(M, e^{2u_\epsilon}g)}. \end{aligned}$$

By Hölder's inequality, we deduce that

$$\int_{B_g(x, r_0)} \psi_\epsilon^2 e^{2u_\epsilon} dv_g \left(1 - \int_{B_g(x, r_0)} e^{2u_\epsilon} dv_g \right) \leq \frac{\lambda_*(B_g(x, r_0), e^{2u_\epsilon} g)}{\lambda_1(M, e^{2u_\epsilon} g)}.$$

By claim 5, there exists $r_0 > 0$ such that

$$\forall \epsilon > 0, \forall x \in M, \int_{B_g(x, r_0)} e^{2u_\epsilon} dv_g \leq \frac{1}{2}$$

and (3.14) follows for this r_0 .

Let's fix

$$r_1 = \frac{1}{C_0} \min \{ \delta, r_0 \}.$$

Then, for any $i \in \{1, \dots, N\}$, any $x \in \omega_i$, by (3.11) and (3.14), we have that

$$\lambda_*(B_{g_i}(x, r_1), e^{2u_\epsilon} g) \geq \lambda_*(B_g(x, C_0 r_1), e^{2u_\epsilon} g) \geq 4\pi.$$

Writing thanks to (3.13) that

$$\int_{B_{g_i}(x, r)} e^{2u_\epsilon} dv_g \leq \frac{\text{Cap}_2(B_{g_i}(x, r), B_{g_i}(x, r_1))}{\lambda_*(B_{g_i}(x, r_1), e^{2u_\epsilon} g)}$$

for all $0 < r < r_1$ and thanks to the fact that g_i is isometric to the Euclidean metric that

$$\text{Cap}_2(B_{g_i}(x, r), B_{g_i}(x, r_1)) = \frac{2\pi}{\ln \frac{r_1}{r}},$$

we get that

$$\int_{B_{g_i}(x, r)} e^{2u_\epsilon} dv_g \leq \frac{1}{2 \ln \frac{r_1}{r}}$$

for all $0 < r < r_1$. This clearly leads to the conclusion of the claim. \diamond

We now focus on the eigenfunctions associated to the first non-zero eigenvalue of such a sequence of metrics. We will prove that the nodal sets of such eigenfunctions can not concentrate to a point :

Claim 7. *There exists $\delta_1 > 0$ such that for any $\epsilon > 0$, any $f \in E_1(e^{2u_\epsilon} g)$ and any $x \in M$,*

$$f(x) = 0 \Rightarrow \exists y \in \partial B_g(x, \delta_1) \text{ s.t. } f(y) = 0.$$

Proof. Assume by contradiction that for any $j \in \mathbb{N}$, there exist $\epsilon_j > 0$, $x_j \in M$ and $f_j \in E_1(e^{2u_{\epsilon_j}} g)$ such that

$$f_j(x_j) = 0 \text{ and } \forall y \in \partial B_g(x_j, 2^{-j}), f_j(y) \neq 0. \quad (3.15)$$

Then, by the maximum principle, f_j changes sign in $B_g(x_j, 2^{-j})$. By the Courant nodal theorem (see [23]), there are two connected nodal domains D_j^1 and D_j^2 for f_j . We know that for $m \in$

$\{1, 2\}$, $D_j^m \cap B_{2^{-j}}(x_j) \neq \emptyset$. Therefore, thanks to (3.15) and what we just said, there is one of this nodal domain, let's say D_j^1 , which satisfies $D_j^1 \subset B_{2^{-j}}(x_j)$.

Then f_j is an eigenfunction on $(D_j^2, e^{2u_{\epsilon_j}})$ with Dirichlet boundary conditions. Since $f_j > 0$, it is an eigenfunction for $\lambda_\star(D_j^2, e^{2u_{\epsilon_j}}) = \lambda_{\epsilon_j}$. Up to a subsequence, $x_j \rightarrow x \in M$ as $j \rightarrow \infty$. Thanks to claim 5, there is an open neighborhood of $\{x\}$, B , with

$$\int_B e^{2u_{\epsilon_j}} \leq \frac{1}{2}.$$

Since $\text{Cap}_2(\{x\}, B) = 0$, we can find $\psi \in C_c^\infty(M \setminus \{x\})$ such that $\psi = 1$ on $M \setminus B$ and

$$\int_M |\nabla \psi|_g^2 dv_g \leq 2\pi.$$

We write then that

$$\lambda_{\epsilon_j} = \lambda_\star(D_j^2, e^{2u_{\epsilon_j}}) \leq \frac{\int_{D_j^2} |\nabla \psi|_g^2 dv_g}{\int_{D_j^2} \psi^2 e^{2u_{\epsilon_j}} dv_g} \leq 4\pi$$

which contradicts (3.12). This ends the proof of the claim. \diamond

At last, the non-concentration estimates just obtained gives the $W^{1,2}(M)$ -boundedness of a sequence of normalized eigenfunctions :

Claim 8. Any sequence of eigenfunctions $f_\epsilon \in E_1(e^{2u_\epsilon}g)$ such that $\int_M f_\epsilon^2 e^{2u_\epsilon} dv_g = 1$ is bounded in $W^{1,2}(M)$.

Proof. We already know that $\int_M |\nabla f_\epsilon|_g^2 dv_g = \lambda_1(e^{2u_\epsilon}g)$ is bounded. We now prove that $\{e^{2u_\epsilon} dv_g\}_\epsilon$ is a bounded sequence in $W^{-1,2}(M) = W^{1,2}(M)^\star$.

Let us consider a finite covering of M by balls $B_g(y_j, r_0)$, $j = 1, \dots, L$ where $r_0 > 0$ is given by (3.14) and let (ψ_j) be a partition of unity associated to this covering. For $\psi \in W^{1,2}(M)$, we have that

$$\begin{aligned} \int_M \psi e^{2u_\epsilon} dv_g &= \sum_{j=1}^L \int_{B_g(y_j, r_0)} \psi \psi_j e^{2u_\epsilon} dv_g \\ &\leq \sum_{j=1}^L \left(\int_{B_g(y_j, r_0)} (\psi_j \psi)^2 e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}} \left(\int_{B_g(y_j, r_0)} e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}} \\ &\leq \sum_{j=1}^L \frac{1}{\lambda_\star(B_g(y_j, r_0), e^{2u_\epsilon}g)^{\frac{1}{2}}} \left(\int_M |\nabla(\psi_j \psi)|^2 dv_g \right)^{\frac{1}{2}} \\ &\leq D_0 \|\psi\|_{W^{1,2}(M)} \end{aligned}$$

where D_0 is independent of ψ and ϵ . Then $\{e^{2u_\epsilon} dv_g\}_\epsilon$ is a bounded sequence in $W^{-1,2}(M)$ and we get the following Poincaré inequality (see [116], lemma 4.1.3) : there exists $D_2 > 0$ such that

$$\forall \epsilon > 0, \forall \psi \in C^\infty(M), \int_M \left(\psi - \int_M \psi e^{2u_\epsilon} dv_g \right)^2 dv_g \leq D_2 \int_M |\nabla \psi|_g^2 dv_g.$$

We apply this inequality to $\psi = f_\epsilon$ which has zero mean value with respect to $e^{2u_\epsilon} dv_g$ since f_ϵ is a first eigenfunction with respect to $e^{2u_\epsilon} g$. We get that

$$\int_M f_\epsilon^2 dv_g \leq D_2 \int_M |\nabla f_\epsilon|_g^2 dv_g = D_2 \lambda_1 (e^{2u_\epsilon} g)$$

which gives the desired conclusion. \diamond

3.3.2 Construction of a maximizing sequence

For $\epsilon > 0$ and $x, y \in M$, we denote by $p_\epsilon(x, y)$ the heat kernel of (M, g) at time ϵ . We let $\mathcal{M}(M)$ be the set of positive Radon measures provided with the weak* topology and $\mathcal{M}_1(M)$ be the subset of probability measures. For $\nu \in \mathcal{M}(M)$, $f \in L^1(M, g)$ and $\epsilon > 0$, we set

$$K_\epsilon[\nu](x) = \int_M p_\epsilon(x, y) d\nu(y)$$

and

$$K_\epsilon[f](x) = \int_M p_\epsilon(x, y) f(y) dv_g(y)$$

so that

$$\int_M K_\epsilon[f](x) d\nu(x) = \int_M f(x) K_\epsilon[\nu] dv_g(x).$$

We refer to [6] for standard properties of the heat operator on Riemannian manifolds.

Let us now define the maximizing sequence we will consider. For $\epsilon > 0$, we set

$$\lambda_\epsilon = \sup_{\nu \in \mathcal{M}_1(M)} \lambda_1(K_\epsilon[\nu]g). \quad (3.16)$$

Since $K_\epsilon[\nu] > 0$ and $K_\epsilon[\nu] \in C^\infty$, $\lambda_\epsilon \leq \Lambda_1(M, [g])$. Moreover, since $K_\epsilon : \mathcal{M}_1(M) \mapsto \mathcal{C}^k(M)$ is continuous for all $k \geq 0$, every maximizing sequence for λ_ϵ converges in $\mathcal{M}_1(M)$. So let $\nu_\epsilon \in \mathcal{M}_1(M)$ be such that

$$\lambda_\epsilon = \lambda_1(K_\epsilon[\nu_\epsilon]g).$$

We claim that

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \Lambda_1(M, [g]). \quad (3.17)$$

We already know that $\lambda_\epsilon \leq \Lambda_1(M, [g])$ for all $\epsilon > 0$. Let $\eta > 0$ and pick up $\tilde{g} \in [g]$ such that $\text{Vol}_{\tilde{g}}(M) = 1$ and $\lambda_1(\tilde{g}) > \Lambda_1(M, [g]) - \frac{\eta}{2}$. Let $u \in C^\infty(M)$ such that $\tilde{g} = e^{2u}g$. By definition of the heat operator, $K_\epsilon[dv_{\tilde{g}}] = K_\epsilon[e^{2u}]$ and we know that $K_\epsilon[e^{2u}] \rightarrow e^{2u}$ in $C^0(M)$ as $\epsilon \rightarrow 0$ (use the uniform estimates (3.28) given in [6]). Then, there exists $\epsilon_0 > 0$ such that

$$\lambda_\epsilon \geq \lambda_1(K_\epsilon[dv_{\tilde{g}}]g) \geq \lambda_1(\tilde{g}) - \frac{\eta}{2} \geq \Lambda_1(M, [g]) - \eta$$

for all $0 < \epsilon < \epsilon_0$. This proves (3.17).

We let in the following

$$K_\epsilon[\nu_\epsilon] = e^{2u_\epsilon}$$

and we have that

$$\lambda_\epsilon = \lambda_1(e^{2u_\epsilon}g) \rightarrow \Lambda_1(M, [g]) \text{ as } \epsilon \rightarrow 0.$$

Let us exploit the fact that ν_ϵ solves the maximization problem (3.16) :

Claim 9. Let's fix $\epsilon > 0$. Then there exist $(\phi_\epsilon^1, \dots, \phi_\epsilon^{k(\epsilon)}) \in \mathcal{C}^\infty(M, \mathbb{R}^k)$ such that

- $\forall i \in \{1, \dots, k(\epsilon)\}, \Delta_g \phi_\epsilon^i = \lambda_\epsilon e^{2u_\epsilon} \phi_\epsilon^i$.
- $\int_M e^{2u_\epsilon} |\Phi_\epsilon|^2 dv_g = 1$.
- $K_\epsilon \left[|\Phi_\epsilon|^2 \right] \geq 1$ on M .
- $K_\epsilon \left[|\Phi_\epsilon|^2 \right] = 1$ on $\text{supp}(v_\epsilon)$.

Here $\Phi_\epsilon = (\phi_\epsilon^1, \dots, \phi_\epsilon^{k(\epsilon)})$ and $|\Phi_\epsilon|^2 = \sum_{i=1}^{k(\epsilon)} (\phi_\epsilon^i)^2$.

Proof. Since ϵ is fixed, up to the end of the proof of this claim, we omit the ϵ indices of λ_ϵ , v_ϵ and e^{2u_ϵ} .

Let $\mu \in \mathcal{M}(M)$ and $t > 0$. We let

$$\lambda_t = \lambda_1(K_\epsilon[v + t\mu]g) .$$

Note that $\lambda = \lambda_{t=0}$. By continuity, $\lambda_t \rightarrow \lambda$ as $t \rightarrow 0^+$. We first prove that

$$\lim_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{t} = \inf_{\phi \in E_1(e^{2u}g)} \left(-\lambda \frac{\int_M K_\epsilon[\phi^2] d\mu}{\int_M \phi^2 e^{2u} dv_g} \right) . \quad (3.18)$$

We let $\phi \in E_1(K_\epsilon[v]g) = E_1(e^{2u}g)$ and we write that

$$\begin{aligned} \lambda_t \left(\int_M \left(\phi - \frac{\int_M \phi K_\epsilon[v + t\mu] dv_g}{\int_M K_\epsilon[v + t\mu] dv_g} \right)^2 K_\epsilon[v + t\mu] dv_g \right) &\leq \int_M |\nabla \phi|_g^2 dv_g \\ &= \lambda \int_M e^{2u} \phi^2 dv_g . \end{aligned}$$

Since $K_\epsilon[v + t\mu] = e^{2u} + tK_\epsilon[\mu]$, we easily get that

$$\lambda_t \left(\int_M \phi^2 e^{2u} dv_g + t \int_M \phi^2 K_\epsilon[\mu] dv_g + o(t) \right) \leq \lambda \int_M e^{2u} \phi^2 dv_g$$

so that

$$\begin{aligned} \frac{\lambda_t - \lambda}{t} &\leq -\lambda \frac{\int_M \phi^2 K_\epsilon[\mu] dv_g}{\int_M \phi^2 e^{2u} dv_g} + o(1) \\ &= -\lambda \frac{\int_M K_\epsilon[\phi^2] d\mu}{\int_M \phi^2 e^{2u} dv_g} + o(1) . \end{aligned}$$

So far we have proved that

$$\limsup_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{t} \leq \inf_{\phi \in E_1(e^{2u}g)} \left(-\lambda \frac{\int_M K_\epsilon[\phi^2] d\mu}{\int_M \phi^2 e^{2u} dv_g} \right) . \quad (3.19)$$

Let now $\phi_t \in E_1(K_\epsilon[v + t\mu]g)$ with $\|\phi_t\|_{L^2(K_\epsilon[v+t\mu]g)} = 1$. We have that

$$\Delta_g \phi_t = \lambda_t K_\epsilon[v + t\mu] \phi_t = \lambda_t (e^{2u} + tK_\epsilon[\mu]) \phi_t . \quad (3.20)$$

For all $t > 0$ small enough, $L^2(K_\epsilon[v + t\mu]g)$ and $L^2(e^{2u}g)$ are the same sets and define equivalent norms and the constants in the equivalence are independent of t . Indeed, we have

$$1 \leq \frac{K_\epsilon[v + t\mu]}{K_\epsilon[v]} \leq 1 + C_\epsilon^2 t \int_M d\mu,$$

where $C_\epsilon > 1$ is a constant such that $C_\epsilon^{-1} \leq p_\epsilon \leq C_\epsilon$ and for $t < (C_\epsilon^2 \int_M d\mu)^{-1}$, we get

$$\forall \phi \in L^2(e^{2u}g), \int_M \phi^2 K_\epsilon[v] dv_g \leq \int_M \phi^2 K_\epsilon[v + t\mu] dv_g \leq 2 \int_M \phi^2 K_\epsilon[v] dv_g.$$

Then, up to the extraction of a subsequence, by standard elliptic theory (see [43] for Sobolev embeddings : Corollary 7.11, page 158, and for elliptic estimates : Theorem 9.11 page 235), there exists $\phi \in E_1(e^{2u}g)$ such that $\phi_t \rightarrow \phi$ in C^m as $t \rightarrow 0^+$ and $\|\phi\|_{L^2(e^{2u}g)} = 1$. We denote by Π the orthogonal projection on $E_1(e^{2u}g)$ with respect to the $L^2(e^{2u}g)$ -norm. Then we rewrite (3.20) as

$$\Delta_g \left(\frac{\phi_t - \Pi\phi_t}{\alpha_t} \right) - \lambda e^{2u} \frac{\phi_t - \Pi\phi_t}{\alpha_t} = \frac{\lambda_t - \lambda}{\alpha_t} e^{2u} \phi_t + \frac{t}{\alpha_t} \lambda_t K_\epsilon[\mu] \phi_t \quad (3.21)$$

where

$$\alpha_t = \|\phi_t - \Pi\phi_t\|_\infty + t + (\lambda - \lambda_t).$$

Then, up to the extraction of a subsequence we have that

$$t_0 = \lim_{t \rightarrow 0^+} \frac{t}{\alpha_t} \text{ and } \delta_0 = \lim_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{\alpha_t}.$$

By standard elliptic theory (see the theorems above of [43]), up to the extraction of a subsequence,

$$\frac{\phi_t - \Pi\phi_t}{\alpha_t} \rightarrow R_0 \text{ in } C^2(M) \text{ as } t \rightarrow 0^+ \quad (3.22)$$

where $R_0 \in E_1(e^{2u}g)^\perp$. Passing to the limit in equation (3.21), we get that

$$\Delta_g R_0 - \lambda e^{2u} R_0 = \delta_0 e^{2u} \phi + t_0 \lambda K_\epsilon[\mu] \phi \quad (3.23)$$

and

$$\|R_0\|_\infty + t_0 + \delta_0 = 1. \quad (3.24)$$

Testing (3.23) against ϕ and using the fact that $R_0 \in E_1(e^{2u}g)^\perp$ give that

$$\delta_0 \int_M e^{2u} \phi^2 dv_g = -t_0 \lambda \int_M K_\epsilon[\mu] \phi^2 dv_g = -t_0 \lambda \int_M K_\epsilon[\phi^2] d\mu.$$

If $t_0 = 0$, then $\delta_0 = 0$ and then $R_0 \equiv 0$ thanks to (3.23) and the fact that $R_0 \in E_1(e^{2u}g)^\perp$. This is absurd thanks to (3.24). Thus $t_0 \neq 0$ and

$$\lim_{t \rightarrow 0^+} \frac{\lambda_t - \lambda}{t} = \frac{\delta_0}{t_0} = -\lambda \frac{\int_M K_\epsilon[\phi^2] d\mu}{\int_M e^{2u} \phi^2 dv_g}.$$

This together with (3.19) gives (3.18).

Now, with a renormalization, $(1 + t \int_M d\mu) \lambda_t \leq \lambda$ for all $t \geq 0$ and we deduce from (3.18) that

$$\forall \mu \in \mathcal{M}(M), \exists \phi \in E_1(e^{2u}g) \text{ s.t. } \int_M \phi^2 e^{2u} dv_g = 1 \text{ and } \int_M (1 - K_\epsilon[\phi^2]) d\mu \leq 0. \quad (3.25)$$

Let us define the following subsets of $\mathcal{C}^0(M)$:

$$K = \left\{ \psi \in \mathcal{C}^0(M); \exists \phi_1, \dots, \phi_k \in E_1(e^{2u}g), \psi = \sum_{i=1}^k K_\epsilon[\phi_i^2] - 1, \int_M \psi dv = 0 \right\}$$

and

$$F = \{f \in \mathcal{C}^0(M); f \geq 0\}.$$

The set F is closed and convex. The set K is clearly convex since it is the translation of the convex hull of

$$C = \left\{ K_\epsilon[\phi^2]; \phi \in E_1(e^{2u}g), \|\phi\|_{L^2(e^{2u}g)} = 1 \right\}.$$

Since E_1 is finite-dimensional, the vector space spanned by C is finite-dimensional and C is compact. Caratheodory's theorem gives that K is also compact.

If $F \cap K = \emptyset$, by Hahn-Banach theorem, there exists $\mu \in \mathcal{M}(M)$ such that

$$\forall f \in F, \int_M f d\mu \geq 0 \quad (3.26)$$

and

$$\forall \psi \in K, \int_M \psi d\mu < 0. \quad (3.27)$$

Then, μ is a non-zero (by (3.27)) positive (by (3.26)) measure and for this measure, (3.27) contradicts (3.25).

Thus $F \cap K \neq \emptyset$ and there exists $\phi^1, \dots, \phi^k \in E_1(e^{2u}g)$ with

$$\int_M |\Phi|^2 e^{2u} dv_g = 1 \text{ and } K_\epsilon[|\Phi|^2] \geq 1$$

where $\Phi = (\phi^1, \dots, \phi^k)$. Moreover, we can write that

$$1 = \int_M |\Phi|^2 e^{2u} dv_g = \int_M K_\epsilon[|\Phi|^2] dv \geq \int_M dv = 1.$$

Therefore, $K_\epsilon[|\Phi|^2] = 1$ ν -a.e. and since $K_\epsilon[|\Phi|^2]$ is continuous, $K_\epsilon[|\Phi|^2] = 1$ on $\text{supp}(\nu)$. This ends the proof of the claim. \diamond

Note that using Gauss's reduction of quadratic forms, we can choose an independant family $(\phi_\epsilon^1, \dots, \phi_\epsilon^{k(\epsilon)})$ in $E_1(e^{2u_\epsilon})$ for which the claim 9 remains true. The number of eigenfunctions $k(\epsilon)$ depends on ϵ but since the multiplicity of eigenvalues is bounded by a constant which only depends on the topology of the surface (see [15]), up to the extraction of a subsequence, we assume in the following that k is fixed.

3.3.3 Estimates on eigenfunctions

We assume now that (M, g) is not diffeomorphic to the sphere. Then, by the rigidity result proved in section 3.2, $\Lambda_1(M, [g]) > 8\pi$ and we can use the non-concentration estimates of the specific maximizing sequence $\{e^{2u_\epsilon}g\}$ of the previous sections. We denote by ν the weak* limit of $\{e^{2u_\epsilon}dv_g\}_{\epsilon>0}$. Then ν is also the weak* limit of $\{dv_\epsilon\}$. Indeed, for $\zeta \in C^0(M)$,

$$\begin{aligned} \left| \int_M \zeta (dv_\epsilon - e^{2u_\epsilon}dv_g) \right| &= \left| \int_M \zeta (dv_\epsilon - K_\epsilon[v_\epsilon]dv_g) \right| \\ &= \left| \int_M (\zeta - K_\epsilon[\zeta]) dv_\epsilon \right| \leq \|\zeta - K_\epsilon[\zeta]\|_\infty \xrightarrow{\epsilon \rightarrow 0} 0. \end{aligned}$$

We aim at proving that ν is absolutely continuous with respect to dv_g with a smooth density.

Up the end of the proof, we get finer and finer estimates on the sequence of eigenfunctions given by claim 9. For that purpose, we shall use the uniform estimates of the heat kernel p_ϵ on M as $\epsilon \rightarrow 0$ (see [6])

$$p_\epsilon(x, y) = \frac{1}{4\pi\epsilon} e^{-\frac{d_g(x, y)^2}{4\epsilon}} (a_0(x, y) + \epsilon a_1(x, y) + \epsilon^2 a_2(x, y) + \dots), \quad (3.28)$$

where a_0, a_1, a_2, \dots are Riemannian invariants in $C^\infty(M \times M)$ such that $a_0(x, x) = 1$. We also have a uniform bound : there exists $A_0 > 0$ such that for all $\epsilon > 0$,

$$\forall x, y \in M, p_\epsilon(x, y) \leq \frac{A_0}{4\pi\epsilon} e^{-\frac{d_g(x, y)^2}{4\epsilon}}. \quad (3.29)$$

Claim 10. For all $i \in \{1, \dots, k\}$, the sequence $\{\phi_\epsilon^i\}_\epsilon$ is uniformly bounded.

Proof. Let $i \in \{1, \dots, k\}$. For all sequence (x_ϵ) of points in M , up to the extraction of a subsequence, we can assume that $x_\epsilon \in \omega_l$ for some l fixed and we denote in the following

$$\tilde{\phi}_\epsilon = \phi_\epsilon^i \left(\exp_{g_l, x_l}(x) \right)$$

so that

$$\Delta_{\tilde{g}} \tilde{\phi}_\epsilon = \lambda_\epsilon e^{2\tilde{u}_\epsilon} \tilde{\phi}_\epsilon \text{ in } B_0(2\delta)$$

with

$$\tilde{u}_\epsilon = (u_\epsilon - v_l) \left(\exp_{g_l, x_l}(x) \right).$$

We also let in the following

$$\tilde{x}_\epsilon = \exp_{g_l, x_l}^{-1}(x_\epsilon).$$

Step 1 - We prove that for all $R > 0$, there exists a constant $C_R > 0$ such that for all sequence (x_ϵ) of points in M with $d_g(x_\epsilon, \text{supp}(\nu_\epsilon)) \leq R\sqrt{\epsilon}$, we have

$$\forall \epsilon > 0, \left| \phi_\epsilon^i(x_\epsilon) \right| \leq C_R.$$

We let

$$\hat{\phi}_\epsilon(x) = \tilde{\phi}_\epsilon(\sqrt{\epsilon}x + \tilde{x}_\epsilon)$$

for $x \in \mathbb{D}_{\delta\sqrt{\epsilon}}$. Then

$$\Delta_{\tilde{g}} \hat{\phi}_{\epsilon} = \epsilon \lambda_{\epsilon} e^{2\tilde{u}_{\epsilon}(\sqrt{\epsilon}x + \tilde{x}_{\epsilon})} \hat{\phi}_{\epsilon}$$

in $\mathbb{D}_{\delta\sqrt{\epsilon}}$. By (3.28), (ϵp_{ϵ}) is uniformly bounded so that $(\epsilon e^{2\tilde{u}_{\epsilon}(\sqrt{\epsilon}x + \tilde{x}_{\epsilon})})$ is uniformly bounded.

Now, let $y_{\epsilon} \in \text{supp}(\nu_{\epsilon})$ such that $d_g(x_{\epsilon}, y_{\epsilon}) \leq R\sqrt{\epsilon}$. Thanks to claim 9, we have

$$K_{\epsilon} \left[|\Phi_{\epsilon}|^2 \right] (y_{\epsilon}) = 1 .$$

Let us write then thanks to (3.28) with $a_0(x, x) = 1$, that

$$\begin{aligned} 1 = K_{\epsilon} \left[|\Phi_{\epsilon}|^2 \right] (y_{\epsilon}) &\geq K_{\epsilon} \left[|\phi_{\epsilon}^i|^2 \right] (y_{\epsilon}) \\ &= \int_M p_{\epsilon}(y, y_{\epsilon}) \left(\phi_{\epsilon}^i(y) \right)^2 dv_g(y) \\ &\geq \frac{1}{4\pi\epsilon} e^{-R^2} (1 + o(1)) \int_{B_g(y_{\epsilon}, 2R\sqrt{\epsilon})} \left(\phi_{\epsilon}^i(y) \right)^2 dv_g(y) . \end{aligned}$$

We let $\tilde{y}_{\epsilon} = \frac{1}{\sqrt{\epsilon}} \left(\exp_{g_l, x_l}^{-1}(y_{\epsilon}) - \tilde{x}_{\epsilon} \right)$ so that, up to a subsequence, $\tilde{y}_{\epsilon} \rightarrow y_0$ as $\epsilon \rightarrow 0$ and we deduce from the previous inequality that, for any $\rho > 0$, there exists $D_{\rho} > 0$ such that

$$\int_{\mathbb{D}_{\rho}(y_0)} \hat{\phi}_{\epsilon}^2 dx \leq D_{\rho} .$$

Thus, by the Sobolev embedding $W^{2,2} \subset C^0$ ([43] Corollary 7.11, page 158) and the L^2 elliptic estimate ([43], Theorem 9.11 page 235), it is clear that $\{\hat{\phi}_{\epsilon}\}$ is uniformly bounded in any compact subset of \mathbb{R}^2 . This gives the step 1.

Step 2 - Let (x_{ϵ}) be a sequence of points in M such that $\phi_{\epsilon}^i(x_{\epsilon}) = \sup_M |\phi_{\epsilon}^i|$. Up to change ϕ_{ϵ}^i into $-\phi_{\epsilon}^i$, such a x_{ϵ} does exist. We aim at proving that $\{\phi_{\epsilon}^i(x_{\epsilon})\}$ is a bounded sequence and the claim would follow. We set

$$\delta_{\epsilon} = d_g(x_{\epsilon}, \text{supp}(\nu_{\epsilon})) .$$

We divide the proof into three cases.

CASE 1 - We assume that $\delta_{\epsilon}^{-1} = O(1)$.

Then, by (3.28), $\{e^{2u_{\epsilon}}\}$ is uniformly bounded in $B_g\left(x_{\epsilon}, \frac{\delta_{\epsilon}}{2}\right)$ and by the claim 8, $\{\phi_{\epsilon}^i\}$ is bounded in $L^2(M)$. Thus, by the Sobolev embedding $W^{2,2} \subset C^0$ ([43] Corollary 7.11, page 158) and the L^2 elliptic estimate ([43], Theorem 9.11 page 235), $\{\phi_{\epsilon}^i(x_{\epsilon})\}$ is bounded.

CASE 2 - We assume that $\delta_{\epsilon} = O(\sqrt{\epsilon})$.

By Step 1, $\{\phi_{\epsilon}^i(x_{\epsilon})\}$ is bounded.

CASE 3 - We assume that $\delta_{\epsilon} \rightarrow 0$ and $\frac{\delta_{\epsilon}}{\sqrt{\epsilon}} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

We let

$$\check{\phi}_{\epsilon}(x) = \tilde{\phi}_{\epsilon}(\delta_{\epsilon}x + \tilde{x}_{\epsilon})$$

for $x \in \mathbb{D}_{\delta\delta_\epsilon^{-1}}$. Then

$$\Delta_{\tilde{g}} \check{\phi}_\epsilon = \delta_\epsilon^2 \lambda_\epsilon e^{2\tilde{u}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon)} \check{\phi}_\epsilon$$

in $\mathbb{D}_{\delta\delta_\epsilon^{-1}}$. Let $y_\epsilon \in \text{supp } \nu_\epsilon$ such that $d_g(x_\epsilon, y_\epsilon) = \delta_\epsilon$ and set

$$\tilde{y}_\epsilon = \delta_\epsilon^{-1} \left(\exp_{g_l, x_l}^{-1}(y_\epsilon) - \tilde{x}_\epsilon \right)$$

so that

$$\tilde{y}_\epsilon \rightarrow \tilde{y}_0 \text{ as } \epsilon \rightarrow 0 \quad (3.30)$$

after passing to a subsequence and set $R = |\tilde{y}_0|$. Thanks to Step 1, we know that

$$\check{\phi}_\epsilon(\tilde{y}_\epsilon) = \phi_\epsilon^i(y_\epsilon) = O(1) . \quad (3.31)$$

Thanks to the estimate (3.28) on the heat kernel, we also know that there exist $D_1 > 0$ and $r > 0$ such that

$$\delta_\epsilon^2 e^{2\tilde{u}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon)} \leq D_1 \text{ in } \mathbb{D}_r(0) . \quad (3.32)$$

Assume first that $\check{\phi}_\epsilon$ does not vanish in $\mathbb{D}_{3R}(0)$. Then we can apply Harnack's inequality thanks to (3.32) to get the existence of some $D_2 > 0$ such that

$$\check{\phi}_\epsilon(x) \geq D_2 \check{\phi}_\epsilon(0) \quad (3.33)$$

for all $\epsilon > 0$ and all $x \in \mathbb{D}_{\frac{r}{2}}(0)$. Note here that $\check{\phi}_\epsilon$ is maximal at 0 thanks to the choice of x_ϵ we made. Since $\check{\phi}_\epsilon$ is super-harmonic on $\mathbb{D}_{|\tilde{y}_\epsilon|}(\tilde{y}_\epsilon) \subset \mathbb{D}_{3R}(0)$, we can also write that

$$\check{\phi}_\epsilon(\tilde{y}_\epsilon) \geq \frac{1}{2\pi |\tilde{y}_\epsilon|} \int_{\partial \mathbb{D}_{|\tilde{y}_\epsilon|}(\tilde{y}_\epsilon)} \check{\phi}_\epsilon d\sigma .$$

Keeping only the part of the integral which lies in $\mathbb{D}_{\frac{r}{2}}(0)$ and using (3.33), we clearly get the existence of some $D_3 > 0$ such that

$$\check{\phi}_\epsilon(\tilde{y}_\epsilon) \geq D_3 \check{\phi}_\epsilon(0) .$$

Here we used the assumption that $\check{\phi}_\epsilon > 0$ in $\mathbb{D}_{3R}(0)$. Thanks to (3.31), we conclude in this situation that $\check{\phi}_\epsilon(0) = \phi_\epsilon(x_\epsilon) = O(1)$.

Assume now that $\check{\phi}_\epsilon$ vanishes in $\mathbb{D}_{3R}(0)$. By the claim 7, since $\delta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, $\check{\phi}_\epsilon$ also vanishes on $\partial \mathbb{D}_{4R}(0)$. By Cheng results on the nodal set of eigenfunctions ([15]) and the Courant nodal theorem ([23]), the first eigenfunction $\check{\phi}_\epsilon$ vanishes on a piecewise smooth curve which connects two points $a_\epsilon \in \partial \mathbb{D}_{3R}(0)$ and $b_\epsilon \in \partial \mathbb{D}_{4R}(0)$.

By [116], corollary 4.5.3, there is a Poincaré inequality which bounds the L^2 -norm of a function ψ by the L^2 norm of its gradient with a multiplicative constant bounded by $B_{1,2}(\{\psi = 0\})^{-\frac{1}{2}}$ where $B_{1,2}$ is the Bessel capacity. The Bessel capacity is equivalent to the variational capacity (see [110] Theorem 3.5.2) and we know that the variational capacity of a continuous curve which connects two uniformly distant points is uniformly bounded from below (see [52], pages 95-97).

Therefore, there is a constant C independent of ϵ such that

$$\int_{\mathbb{D}_{4R}(0)} \check{\phi}_\epsilon^2 dx \leq C \int_{\mathbb{D}_{4R}(0)} |\nabla \check{\phi}_\epsilon|^2 dx .$$

By the conformal invariance, the L^2 -norm of the gradient is uniformly bounded. Thus $\{\check{\phi}_\epsilon\}$ is bounded in $L^2\left(\mathbb{D}_{\frac{r}{2}}(0)\right)$. By the Sobolev embedding $W^{2,2} \subset C^0$ ([43] Corollary 7.11, page 158) and the L^2 elliptic estimate ([43], Theorem 9.11 page 235), thanks to (3.32), we get also in this second situation that $\{\phi_\epsilon(x_\epsilon)\}$ is bounded. This ends the proof of the claim. \diamond

We get now a quantitative non-concentration estimate on the L^2 -norm of the gradient of $\phi_\epsilon^i, i = 1, \dots, k$.

Claim 11. *There exists $C_2 > 0$ such that*

$$\int_{B_g(x,r)} |\nabla \Phi_\epsilon|_g^2 dv_g \leq \frac{C_2}{\sqrt{\ln \frac{1}{r}}}$$

for all $\epsilon > 0$ and all $r > 0$. Here

$$|\nabla \Phi_\epsilon|_g^2 = \sum_{i=1}^k |\nabla \phi_\epsilon^i|_g^2.$$

Proof. It is clearly sufficient to prove the result for any r small enough and any $x \in \omega_l, l$ fixed. Thus, setting as above

$$\tilde{\phi}_\epsilon^i = \phi_\epsilon^i \left(\exp_{g_l, x_l}(x) \right)$$

and $\tilde{\Phi}_\epsilon = \left(\tilde{\phi}_\epsilon^i \right)_{i=1, \dots, k}$, we need to prove that, for $r \leq \delta$ and $x \in \mathbb{D}_\delta(0)$,

$$\int_{\mathbb{D}_r(x)} |\nabla \tilde{\Phi}_\epsilon|_\zeta^2 dv_\zeta \leq \frac{C_2}{\sqrt{\ln \frac{1}{r}}}$$

for some $C_2 > 0$. In the following, we shall assume without loss of generality that $\delta < 1$. Let us set

$$F_\epsilon(r) = \int_{\mathbb{D}_r(x)} |\nabla \tilde{\Phi}_\epsilon|_\zeta^2 dv_\zeta.$$

Using the equation satisfied by $\tilde{\Phi}_\epsilon$, namely

$$\Delta_\zeta \tilde{\Phi}_\epsilon = \lambda_\epsilon e^{2u_\epsilon^l} \tilde{\Phi}_\epsilon$$

where

$$u_\epsilon^l = (u_\epsilon - v_l) \left(\exp_{x_l, g_l}(x) \right),$$

we get that

$$F_\epsilon(r) = \lambda_\epsilon \int_{\mathbb{D}_r(x)} e^{2u_\epsilon^l} |\tilde{\Phi}_\epsilon|^2 dv_\zeta + \int_{\partial \mathbb{D}_r(x)} \tilde{\Phi}_\epsilon \cdot \partial_\nu \tilde{\Phi}_\epsilon d\sigma_\zeta.$$

Using now claims 6 and 10, we can write that

$$\begin{aligned} F_\epsilon(r)^2 &\leq \frac{D_1}{\left(\ln \frac{1}{r}\right)^2} + D_2 \left(\int_{\partial \mathbb{D}_r(x)} |\nabla \tilde{\Phi}_\epsilon|_\zeta d\sigma_\zeta \right)^2 \\ &\leq \frac{D_1}{\left(\ln \frac{1}{r}\right)^2} + 2\pi r D_2 \int_{\partial \mathbb{D}_r(x)} |\nabla \tilde{\Phi}_\epsilon|_\zeta^2 d\sigma_\zeta \\ &= \frac{D_1}{\left(\ln \frac{1}{r}\right)^2} + 2\pi r D_2 F'_\epsilon(r). \end{aligned}$$

We can write then that

$$\begin{aligned}
 \left(F_\epsilon(r) \sqrt{\ln \frac{1}{r}} \right)'(s) &= F'_\epsilon(s) \sqrt{\ln \frac{1}{s}} - \frac{1}{2s \sqrt{\ln \frac{1}{s}}} F_\epsilon(s) \\
 &\geq \frac{F_\epsilon(s)^2 \sqrt{\ln \frac{1}{s}}}{2\pi s D_2} - \frac{D_1}{2\pi s D_2 (\ln \frac{1}{s})^{\frac{3}{2}}} - \frac{1}{2s \sqrt{\ln \frac{1}{s}}} F_\epsilon(s) \\
 &\geq -\frac{D_3}{s (\ln \frac{1}{s})^{\frac{3}{2}}}
 \end{aligned}$$

for some $D_3 > 0$ independent of x, ϵ and s . Integrating this inequality from r to δ leads to

$$\begin{aligned}
 F_\epsilon(r) \sqrt{\ln \frac{1}{r}} &\leq F_\epsilon(\delta) \sqrt{\ln \frac{1}{\delta}} + \int_r^\delta \frac{D_3}{s (\ln \frac{1}{s})^{\frac{3}{2}}} ds \\
 &\leq \lambda_\epsilon \sqrt{\ln \frac{1}{\delta}} + \frac{2D_3}{\sqrt{\ln \frac{1}{\delta}}},
 \end{aligned}$$

where by the conformal invariance of the Dirichlet energy

$$F_\epsilon(\delta) = \int_{\mathbb{D}_\delta(x)} |\nabla \tilde{\Phi}_\epsilon|_\xi^2 dv_\xi \leq \int_{\Omega_l} |\nabla \tilde{\Phi}_\epsilon|_g^2 dv_g \leq \lambda_\epsilon.$$

We clearly ended the proof of the claim. \diamond

Thanks to the previous claims 10 and 11, we can compare precisely Φ_ϵ and $K_\epsilon[\Phi_\epsilon]$:

Claim 12. *There exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that*

$$\forall x \in M, |\Phi_\epsilon(x)|^2 \geq 1 - \beta_\epsilon \quad (3.34)$$

and

$$\forall x \in \text{supp}(v_\epsilon), |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon \quad (3.35)$$

Proof. We first prove that there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x, y \in M, d_g(x, y) \leq \frac{\sqrt{\epsilon}}{\beta_\epsilon} \Rightarrow |\Phi_\epsilon(x) - \Phi_\epsilon(y)| \leq \beta_\epsilon. \quad (3.36)$$

For that purpose, let us set

$$\gamma_\epsilon = \|\epsilon e^{2u_\epsilon}\|_\infty^{\frac{1}{3}}.$$

Using claim 6 and (3.28), it is easily seen that $\gamma_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. Indeed, for any $r > 0$,

$$\epsilon e^{2u_\epsilon(x)} \leq (4\pi + o(1)) \int_{B_g(x, r)} dv_\epsilon + o(1) = 4\pi v(B_g(x, r)) + o(1) \leq \frac{4\pi C_1}{\ln \frac{1}{r}} + o(1).$$

We also have that

$$\frac{\gamma_\epsilon}{\sqrt{\epsilon}} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

since $\gamma_\epsilon \geq \epsilon^{\frac{1}{3}}$. Let now $(x_\epsilon, y_\epsilon) \in M^2$ with $d_g(x_\epsilon, y_\epsilon) \leq \frac{\sqrt{\epsilon}}{\gamma_\epsilon}$. Up to the extraction of a subsequence, $x_\epsilon \in \omega_l$ for some l fixed. Let us set as before

$$\tilde{\Phi}_\epsilon = \Phi_\epsilon \left(\exp_{g_l, x_l}(x) \right)$$

which satisfies

$$\Delta_{\tilde{g}} \tilde{\Phi}_\epsilon = \lambda_\epsilon e^{2u_\epsilon^l} \tilde{\Phi}_\epsilon$$

with

$$u_\epsilon^l = (u_\epsilon - v_l) \left(\exp_{g_l, x_l}(x) \right).$$

We set

$$\hat{\Phi}_\epsilon(x) = \tilde{\Phi}_\epsilon \left(\tilde{x}_\epsilon + \frac{\sqrt{\epsilon}}{\gamma_\epsilon} x \right)$$

where $x_\epsilon = \exp_{g_l, x_l}(\tilde{x}_\epsilon)$. We let α_ϵ be the mean value of $\hat{\Phi}_\epsilon$ in $\mathbb{D}_{3C_0}(0)$. Using the Sobolev embedding $W^{2,2} \subset C^0$ ([43], Corollary 7.11, page 158), the L^2 elliptic estimate ([43], Theorem 9.11, page 235) and the Poincaré inequality ([43], Formula (7.45), page 164), we know that there exists $D > 1$ such that

$$\begin{aligned} \|\hat{\Phi}_\epsilon - \alpha_\epsilon\|_{L^\infty(\mathbb{D}_{2C_0}(0))} &\leq D \|\Delta \hat{\Phi}_\epsilon\|_{L^\infty(\mathbb{D}_{3C_0}(0))} + D \|\nabla \hat{\Phi}_\epsilon\|_{L^2(\mathbb{D}_{3C_0}(0))} \\ &\leq D \|\Phi_\epsilon\|_\infty C_0^2 \lambda_\epsilon \gamma_\epsilon + D \frac{\sqrt{C_2}}{\left(\ln \frac{\gamma_\epsilon}{3C_0^2 \sqrt{\epsilon}} \right)^{\frac{1}{4}}} \end{aligned}$$

thanks to claim 11. Setting

$$\beta_\epsilon = 2 \left(D \|\Phi_\epsilon\|_\infty C_0^2 \lambda_\epsilon \gamma_\epsilon + D \frac{\sqrt{C_2}}{\left(\ln \frac{\gamma_\epsilon}{3C_0^2 \sqrt{\epsilon}} \right)^{\frac{1}{4}}} \right),$$

we then get that $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and that

$$|\Phi_\epsilon(x_\epsilon) - \Phi_\epsilon(y_\epsilon)| \leq \beta_\epsilon,$$

which clearly proves (3.36).

We prove now that for all sequence $\{f_\epsilon\}$ of uniformly bounded functions which satisfy

$$\forall x, y \in M, d_g(x, y) \leq \frac{\sqrt{\epsilon}}{\beta_\epsilon} \Rightarrow |f_\epsilon(x) - f_\epsilon(y)| \leq \beta_\epsilon, \quad (3.37)$$

then, up to increase β_ϵ , we have that

$$\forall x \in M, |f_\epsilon(x) - K_\epsilon[f_\epsilon](x)| \leq \beta_\epsilon. \quad (3.38)$$

Indeed, for $x \in M$,

$$\begin{aligned} |f_\epsilon - K_\epsilon[f_\epsilon]|(x) &\leq \int_{B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} |f_\epsilon(x) - f_\epsilon(y)| p_\epsilon(x, y) dv_g(y) \\ &\quad + \int_{M \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} |f_\epsilon(x) - f_\epsilon(y)| p_\epsilon(x, y) dv_g(y). \end{aligned}$$

By the property (3.37),

$$|f_\epsilon(x) - K_\epsilon[f_\epsilon](x)| \leq \beta_\epsilon + 2 \|f_\epsilon\|_\infty \int_{M \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x, y) dv_g(y)$$

and with (3.29) and (3.11),

$$\begin{aligned} \int_{M \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x, y) dv_g(y) &\leq \frac{A_0}{4\pi\epsilon} \int_{M \setminus \Omega_l} e^{-\frac{d_g(x, y)^2}{4\epsilon}} dv_g(y) \\ &\quad + \frac{A_0}{4\pi\epsilon} \int_{\Omega_l \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} e^{-\frac{d_g(x, y)^2}{4\epsilon}} dv_g(y) \\ &\leq \frac{A_0}{4\pi\epsilon} e^{-\frac{\delta^2}{4\epsilon C_0^2}} + \frac{A_0 C_0^2}{4\pi\epsilon} \int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{\sqrt{\epsilon}}{C_0 \beta_\epsilon}}} e^{-\frac{|y|^2}{4C_0^2 \epsilon}} dy \\ &\leq O\left(\frac{e^{-\frac{\delta^2}{4\epsilon C_0^2}}}{\epsilon}\right) + \frac{A_0 C_0^4}{4\pi} \int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{1}{C_0^2 \beta_\epsilon}}} e^{-\frac{|z|^2}{4}} dz \\ &\leq O\left(\frac{e^{-\frac{\delta^2}{4\epsilon C_0^2}}}{\epsilon}\right) + O\left(e^{-\frac{1}{4C_0^4 \beta_\epsilon^2}}\right). \end{aligned}$$

Up to increase β_ϵ , we get (3.38).

Up to increase β_ϵ , we get (3.38) for $f_\epsilon = |\Phi_\epsilon|^2$, thanks to (3.36). Then, by claim 9, we easily get (3.34). By claim 9, we also have that

$$\forall x \in \text{supp}(v_\epsilon), \left| |\Phi_\epsilon(x)|^2 - 1 \right| \leq \beta_\epsilon \quad (3.39)$$

Again, up to increase β_ϵ , we get (3.38) for $f_\epsilon = |\Phi_\epsilon|$ thanks to (3.36). Then, by (3.39), we easily get (3.35). This ends the proof of the claim. \diamond

Thanks to claim 12, we can define $\Psi_\epsilon = \frac{\Phi_\epsilon}{|\Phi_\epsilon|} \in C^\infty(M, \mathbb{S}^{k-1})$. Then, thanks to claim 8, $\{\Psi_\epsilon\}_\epsilon$ is bounded in $W^{1,2}(M, \mathbb{S}^{k-1})$.

Claim 13. *There exists $C_3 > 0$ such that*

$$|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2 \sqrt{\ln \frac{2\delta(M)}{d_g(x, y)}} \leq C_3$$

for all $x, y \in M$ and all $\epsilon > 0$ where $\delta(M)$ is the diameter of M . In particular, the sequence $\{\Psi_\epsilon\}_\epsilon$ is uniformly equicontinuous in $C^0(M, \mathbb{S}^{k-1})$.

Proof. We first claim that there exists $D_1 > 0$ such that

$$\sup_{x \in M} \sup_{v \in \Psi_\epsilon(x)^\perp \cap \mathbb{S}^{k-1}} \frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq \frac{D_1}{\sqrt{\ln \frac{1}{r}}} \quad (3.40)$$

for all r small enough and all $\epsilon > 0$.

For $x \in M$ and $v \in \Psi_\epsilon(x)^\perp \cap \mathbb{S}^{k-1}$, the eigenfunction $\Phi_\epsilon \cdot v$ vanishes at x . Using claim 7, we can argue as in the proof of claim 10 to get the existence of some $D_2 > 0$ such that

$$\frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq D_2 \int_{B_g(x, r)} |\nabla (\Phi_\epsilon \cdot v)|_g^2 dv_g$$

for all r small enough. We deduce thanks to claim 11 that

$$\frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq \frac{D_2 C_2}{\sqrt{\ln \frac{1}{r}}}$$

for all r small enough and (3.40) follows.

Assume now by contradiction that the conclusion of the claim is false, that is there exists $\epsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, x_n and y_n in M such that

$$|\Psi_{\epsilon_n}(x_n) - \Psi_{\epsilon_n}(y_n)|^2 \sqrt{\ln \frac{1}{r_n}} \rightarrow +\infty \text{ as } n \rightarrow +\infty \quad (3.41)$$

where $r_n = d_g(x_n, y_n) \rightarrow 0$ as $n \rightarrow +\infty$. Thanks to (3.34) of claim 12, up to the extraction of a subsequence, there exists a fixed $v \in \mathbb{S}^{k-1}$ such that

$$\frac{1}{\text{Vol}_g(B_g(x_n, r_n))} \int_{B_g(x_n, r_n)} (\Phi_{\epsilon_n} \cdot v)^2 dv_g \geq \frac{1 - \beta_{\epsilon_n}}{k} = \frac{1}{k} + o(1).$$

Thanks to the hypothesis (3.41), we now prove that there exists $X_n \in \Psi_{\epsilon_n}(x_n)^\perp$ and $Y_n \in \Psi_{\epsilon_n}(y_n)^\perp$ such that

$$v = X_n + Y_n \text{ and } |X_n|^2 + |Y_n|^2 = o\left(\sqrt{\ln \frac{1}{r_n}}\right). \quad (3.42)$$

We denote $a_n = \Psi_{\epsilon_n}(x_n) \in \mathbb{S}^{k-1}$, $b_n = \Psi_{\epsilon_n}(y_n) \in \mathbb{S}^{k-1}$ and Π_n the plane generated by a_n and b_n . Let $c_n \in \Pi_n \cap \mathbb{S}^{k-1}$ such that $\{a_n, c_n\}$ is an orthonormal basis of Π_n . We get $\theta_n \in \mathbb{R}$ such that

$$b_n = \cos \theta_n a_n + \sin \theta_n c_n.$$

Then $v = p_n + q_n$ with $p_n \in \Pi_n$ and $q_n \in \Pi_n^\perp$. Notice that $|p_n| \leq 1$ and $|q_n| \leq 1$. Let $\alpha_n \in \mathbb{R}$ such that

$$p_n = |p_n| (\cos \alpha_n a_n + \sin \alpha_n c_n).$$

We then set

$$X_n = t_n c_n + q_n \in a_n^\perp$$

$$Y_n = s_n (-\sin \theta_n a_n + \cos \theta_n c_n) \in b_n^\perp$$

with

$$s_n = -|p_n| \frac{\cos \alpha_n}{\sin \theta_n}$$

$$t_n = |p_n| \left(\sin \alpha_n + \frac{\cos \alpha_n \cos \theta_n}{\sin \theta_n} \right)$$

so that $v = X_n + Y_n$. Then,

$$|X_n|^2 + |Y_n|^2 = |q_n|^2 + t_n^2 + s_n^2 \leq 1 + f_{\theta_n}(\alpha_n),$$

where for α and $\theta \in \mathbb{R}$,

$$f_{\theta}(\alpha) = \frac{\cos^2 \alpha}{\sin^2 \theta} + \left(\sin \alpha + \frac{\cos \alpha \cos \theta}{\sin \theta} \right)^2 = \frac{1 + \cos^2 \theta \cos 2\alpha + \cos \theta \sin \theta \sin 2\alpha}{\sin^2 \theta}.$$

We easily prove that $f_{\theta}(\alpha) \leq f_{\theta}(\frac{\theta}{2}) = \frac{1}{1 - \cos \theta}$. Then,

$$|X_n|^2 + |Y_n|^2 \leq O\left(\frac{1}{1 - \cos \theta_n}\right) = O\left(\frac{1}{|a_n - b_n|^2}\right) = o\left(\sqrt{\ln \frac{1}{r_n}}\right).$$

Using (3.40) and (3.42), we write that

$$\begin{aligned} \frac{1}{k} + o(1) &\leq \frac{1}{\text{Vol}_g(B_g(x_n, r_n))} \int_{B_g(x_n, r_n)} (\Phi_{\epsilon_n} \cdot v)^2 dv_g \\ &\leq \frac{2}{\text{Vol}_g(B_g(x_n, r_n))} \int_{B_g(x_n, r_n)} (\Phi_{\epsilon_n} \cdot X_n)^2 dv_g \\ &\quad + \frac{2\text{Vol}_g(B_g(y_n, 2r_n))}{\text{Vol}_g(B_g(x_n, r_n))} \frac{1}{\text{Vol}_g(B_g(y_n, 2r_n))} \int_{B_g(y_n, 2r_n)} (\Phi_{\epsilon_n} \cdot Y_n)^2 dv_g \\ &\leq 2D_1 |X_n|^2 \left(\ln \frac{1}{r_n}\right)^{-\frac{1}{2}} + 8C_0^8 D_1 |Y_n|^2 \left(\ln \frac{1}{2r_n}\right)^{-\frac{1}{2}} \\ &= o(1), \end{aligned}$$

where C_0 satisfies (3.11). This is clearly a contradiction and proves the claim. \diamond

Up to the extraction of a subsequence, one gets functions $\Phi \in W^{1,2}(M, \mathbb{R}^k) \cap L^\infty(M, \mathbb{R}^k)$ and $\Psi \in W^{1,2}(M, \mathbb{S}^{k-1}) \cap \mathcal{C}^0(M, \mathbb{S}^{k-1})$ such that

$$\Phi_\epsilon \rightharpoonup \Phi \text{ in } W^{1,2}(M, \mathbb{R}^k) \text{ and } \Phi_\epsilon \rightarrow \Phi \text{ in } L^p(M, \mathbb{R}^k) \text{ as } \epsilon \rightarrow 0 \quad (3.43)$$

and

$$\Psi_\epsilon \rightharpoonup \Psi \text{ in } W^{1,2}(M, \mathbb{S}^{k-1}) \text{ and } \Psi_\epsilon \rightarrow \Psi \text{ in } \mathcal{C}^0(M, \mathbb{S}^{k-1}) \text{ as } \epsilon \rightarrow 0 \quad (3.44)$$

where Ψ and Φ satisfy

$$|\Phi|^2 \geq_{a.e.} 1 \text{ and } \Psi = \frac{\Phi}{|\Phi|}.$$

Claim 14. For $i \in \{1, \dots, k\}$,

$$\phi_\epsilon^i e^{2u_\epsilon} dv_g \rightharpoonup^* \psi^i dv. \quad (3.45)$$

And, in a weak sense, we have that

$$\Delta_g \phi^i = \Lambda_1(M, [g]) \psi^i dv. \quad (3.46)$$

Proof. Let $\zeta \in C^0(M)$. Then

$$\begin{aligned} \int_M \zeta \phi_\epsilon^i e^{2u_\epsilon} dv_g - \int_M \zeta \psi^i dv &= \int_M \left(K_\epsilon[\zeta \phi_\epsilon^i] - \zeta K_\epsilon[\phi_\epsilon^i] \right) dv_\epsilon \\ &+ \int_M \zeta \left(K_\epsilon[\phi_\epsilon^i] - \psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] \right) dv_\epsilon \\ &+ \int_M \zeta \left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] - \psi_\epsilon^i \right) dv_\epsilon \\ &+ \int_M \zeta \left(\psi_\epsilon^i dv_\epsilon - \psi^i dv \right). \end{aligned}$$

The first term converges to 0 since $\{\phi_\epsilon^i\}$ is uniformly bounded thanks to claim 10. The second term converges to 0 since $(|\Phi_\epsilon|)$ is uniformly bounded thanks to claim 10 and $\{\psi_\epsilon^i\}$ is uniformly equicontinuous thanks to claim 13. The third term converges to 0 thanks to (3.35) (see claim 12). The last term also converges to 0 thanks to the C^0 -convergence of ψ_ϵ^i to ψ^i (see (3.44)) and the weak*-convergence of dv_ϵ to dv . The first part of the claim follows. The second part of the claim is obtained by passing to the weak limit in the equations satisfied by the eigenfunctions thanks to (3.17), (3.43) and (3.45). \diamond

We are now in position to end the proof of theorem 4. We test the equation (3.46) against ψ^i and sum over i to obtain that

$$\sum_{i=1}^k \int_M \langle \nabla \psi^i, \nabla \phi^i \rangle_g dv_g = \Lambda_1(M, [g]) \sum_{i=1}^k \int_M (\psi^i)^2 dv = \Lambda_1(M, [g]).$$

Since

$$\nabla \psi^i = \nabla \left(\frac{\phi^i}{|\Phi|} \right) = \frac{\nabla \phi^i}{|\Phi|} - \frac{\phi^i \nabla |\Phi|}{|\Phi|^2},$$

we deduce that

$$\Lambda_1(M, [g]) = \sum_{i=1}^k \int_M \langle \nabla \psi^i, \nabla \phi^i \rangle_g dv_g = \int_M \left(\frac{|\nabla \Phi|_g^2}{|\Phi|} - \frac{|\nabla(|\Phi|)|_g^2}{|\Phi|} \right) dv_g.$$

Since $\Phi_\epsilon \rightharpoonup \Phi$ in $W^{1,2}(M, \mathbb{R}^k)$ and $|\Phi| \geq_{a.e.} 1$, we have the sequence of inequalities

$$\begin{aligned} \Lambda_1(M, [g]) &= \lim_{\epsilon \rightarrow 0} \int_M |\nabla \Phi_\epsilon|_g^2 dv_g \geq \int_M |\nabla \Phi|_g^2 dv_g \\ &\geq \int_M \frac{|\nabla \Phi|_g^2}{|\Phi|} dv_g \\ &\geq \Lambda_1(M, [g]) + \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g \\ &\geq \Lambda_1(M, [g]). \end{aligned}$$

Thus all the inequalities are in fact equalities and we deduce that $|\Phi| \equiv 1$ so that $\Psi = \Phi$ and that $\Phi_\epsilon \rightarrow \Phi$ in $W^{1,2}(M, \mathbb{R}^k)$ as $\epsilon \rightarrow 0$. We write that

$$0 = \frac{1}{2} \Delta_g (|\Phi|^2) = \sum_{i=1}^k \phi_i \Delta_g \phi_i - \sum_{i=1}^k |\nabla \phi_i|_g^2 = \Lambda_1(M, [g]) |\Phi|^2 dv - |\nabla \Phi|_g^2$$

in a weak sense thanks to (3.46) and what we just said. Then $dv = \frac{|\nabla \Phi|_g^2}{\Lambda_1(M, [g])} dv_g$ and the equation (3.46) becomes

$$\Delta_g \Phi = |\nabla \Phi|_g^2 \Phi$$

with $\Phi \in C^0(M, S^{k-1}) \cap W^{1,2}(M, S^{k-1})$. Such a Φ is called weakly harmonic. By the regularity theory for weakly harmonic maps by Hélein (see [51]), Φ is then smooth and thus harmonic and we can complete the proof of the theorem.

3.4 Existence of maximal metrics for the first eigenvalue

In this section, we prove theorem 5. Since it has already been proved in genus 0 (Hersch [54]) and in genus 1 (Nadirashvili [83]), we prove it for $\gamma \geq 2$. However, our proof clearly works in genus 1 with light modifications (in the description of degeneracy of conformal classes) and this together with the result of El Soufi and Ilias [34] give a new proof of the fact that the flat equilateral torus is maximizing the first eigenvalue of the Laplacian among the tori.

We let M be a smooth compact orientable surface of genus $\gamma \geq 2$ and we let (c_α) be a sequence of conformal classes on M such that

$$\lambda_\alpha = \Lambda_1(M, c_\alpha) \rightarrow \Lambda_1(\gamma) \text{ as } \alpha \rightarrow +\infty. \quad (3.47)$$

Let h_α be the hyperbolic metric of curvature -1 in the conformal class c_α . By theorem 4, we know that there exists $g_\alpha \in c_\alpha$, smooth except at a finite set of conical singularities, such that

$$\text{Vol}_{g_\alpha}(M) = 1 \text{ and } \lambda_1(g_\alpha) = \Lambda_1(M, c_\alpha) = \lambda_\alpha. \quad (3.48)$$

Moreover there exists a smooth harmonic map $\Phi_\alpha : (M, h_\alpha) \mapsto S^{k_\alpha}$ for some $k_\alpha \geq 2$ such that

$$g_\alpha = \frac{|\nabla \Phi_\alpha|_{h_\alpha}^2}{\lambda_\alpha} h_\alpha. \quad (3.49)$$

Since the multiplicity of eigenvalues is bounded by a constant which depends only on the genus γ (see [15]), the sequence (k_α) is uniformly bounded. Up to the extraction of a subsequence, we can assume in the following that k_α is fixed, $k_\alpha \equiv k$ for all α .

The aim is to prove that there exists a family of diffeomorphisms τ_α of M , such that the sequence $(\tau_\alpha^* h_\alpha)$ of hyperbolic metrics does converge smoothly to some hyperbolic metric up to the extraction of a subsequence as $\alpha \rightarrow +\infty$. For that purpose, it suffices to prove that the injectivity radius of h_α does not converge to 0 by Mumford's compactness theorem (see [82]). Then, the sequence of harmonic maps $\Phi_\alpha \circ \tau_\alpha$ converges up to the formation of bubbles which correspond to points of concentration of the measure $dv_{\tau_\alpha^* g_\alpha}$ (see [88], [100] or [115], theorem 2.2). It is clear that claim 5 applies when we allow the reference metric g to lie in a compact set of metrics (here $(\tau_\alpha^* h_\alpha)$) instead of fixing it. Thus, the concentration of the measure $dv_{\tau_\alpha^* g_\alpha}$ (associated to the metric $\tau_\alpha^* g_\alpha$ which is conformal to $\tau_\alpha^* h_\alpha$) cannot occur and theorem 5 would follow.

We proceed by contradiction and assume from now on that

$$i_{h_\alpha}(M) \rightarrow 0 \text{ as } \alpha \rightarrow +\infty. \quad (3.50)$$

Then there exist s closed geodesics $\gamma_\alpha^1, \dots, \gamma_\alpha^s$ whose length l_α^i goes to 0 where $1 \leq s \leq 3\gamma - 3$ (see [55], IV, lemma 4.1). By the collar lemma (see [55], IV, proposition 4.2 or [115], lemma 4.2, for the version we use), for all $1 \leq i \leq s$, there exists an open neighborhood P_α^i of γ_α^i isometric to the following truncated hyperbolic cylinder

$$C_\alpha^i = \left\{ (t, \theta), -\mu_\alpha^i < t < \mu_\alpha^i, 0 \leq \theta < 2\pi \right\} \quad (3.51)$$

with

$$\mu_\alpha^i = \frac{\pi}{l_\alpha^i} \left(\pi - 2 \arctan \left(\sinh \frac{l_\alpha^i}{2} \right) \right) \quad (3.52)$$

endowed with the metric

$$h_\alpha^i = \left(\frac{l_\alpha^i}{2\pi \cos \left(\frac{l_\alpha^i}{2\pi} t \right)} \right)^2 (dt^2 + d\theta^2). \quad (3.53)$$

Note that we identify $\{\theta = 0\}$ with $\{\theta = 2\pi\}$ and that the closed geodesic γ_α^i corresponds to $\{t = 0\}$.

Let us denote by $M_\alpha^1, \dots, M_\alpha^r$ the connected components of $M \setminus \bigcup_{i=1}^s P_\alpha^i$. Then

$$M = \left(\bigcup_{i=1}^s P_\alpha^i \right) \cup \left(\bigcup_{j=1}^r M_\alpha^j \right) \quad (3.54)$$

and this is a disjoint union.

For $0 < b < \mu_\alpha^i$, we let

$$P_\alpha^i(b) = \left\{ (t, \theta), -\mu_\alpha^i + b < t < \mu_\alpha^i - b \right\} \quad (3.55)$$

after identification with C_α^i . We let also $M_\alpha^j(b)$ be the connected component of $M \setminus \bigcup_{i=1}^s P_\alpha^i(b)$

which contains M_α^j . We claim that

Claim 15. *There exists $D > 0$ such that one of the two following assertions is true :*

(a) *There exists $i \in \{1, \dots, s\}$ such that*

$$\text{Vol}_{g_\alpha} \left(P_\alpha^i(a_\alpha) \right) \geq 1 - \frac{D}{a_\alpha}$$

for all sequences $a_\alpha \rightarrow +\infty$ with $\frac{a_\alpha}{\mu_\alpha^i} \rightarrow 0$ as $\alpha \rightarrow +\infty$ for all $1 \leq i \leq s$.

(b) *There exists $j \in \{1, \dots, r\}$ such that*

$$\text{Vol}_{g_\alpha} \left(M_\alpha^j(9a_\alpha) \right) \geq 1 - \frac{D}{a_\alpha}$$

for all sequences $a_\alpha \rightarrow +\infty$ with $\frac{a_\alpha}{\mu_\alpha^i} \rightarrow 0$ as $\alpha \rightarrow +\infty$ for all $1 \leq i \leq s$.

Proof. We first construct test-functions for $\lambda_\alpha = \lambda_1(g_\alpha)$ compactly supported in the hyperbolic cylinders and in the M_α^j 's. We let $b_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$ with $\frac{b_\alpha}{\mu_\alpha^i} \rightarrow 0$ as $\alpha \rightarrow +\infty$ for all $1 \leq i \leq s$.

Test functions in the hyperbolic cylinders.

For $1 \leq i \leq s$, we define φ_α^i as follows. It is 0 outside of P_α^i and on P_α^i , it is defined by

$$\varphi_\alpha^i(t, \theta) = \begin{cases} 0 & \text{for } -\mu_\alpha^i < t \leq -\mu_\alpha^i + 2b_\alpha \\ \frac{\mu_\alpha^i - 2b_\alpha + t}{b_\alpha} & \text{for } -\mu_\alpha^i + 2b_\alpha < t \leq -\mu_\alpha^i + 3b_\alpha \\ 1 & \text{for } -\mu_\alpha^i + 3b_\alpha < t < \mu_\alpha^i - 3b_\alpha \\ \frac{\mu_\alpha^i - 2b_\alpha - t}{b_\alpha} & \text{for } \mu_\alpha^i - 3b_\alpha \leq t < \mu_\alpha^i - 2b_\alpha \\ 0 & \text{for } \mu_\alpha^i - 2b_\alpha \leq t < \mu_\alpha^i \end{cases} \quad (3.56)$$

We clearly have that

$$\int_M |\nabla \varphi_\alpha^i|_{g_\alpha}^2 dv_{g_\alpha} = \int_M |\nabla \varphi_\alpha^i|_{h_\alpha}^2 dv_{h_\alpha} = \frac{4\pi}{b_\alpha} \quad (3.57)$$

and that

$$\int_M (\varphi_\alpha^i)^2 dv_{g_\alpha} \geq \text{Vol}_{g_\alpha}(P_\alpha^i(3b_\alpha)) \quad (3.58)$$

for $1 \leq i \leq s$.

Test functions in the connected components M_α^j .

For $1 \leq j \leq r$, we define ψ_α^j as follows. It is 1 in M_α^j , 0 in all the M_α^k 's, $k \neq j$. And, in the P_α^i 's, it is defined as follows. It is 0 for $-\mu_\alpha^i + 2b_\alpha \leq t \leq \mu_\alpha^i - 2b_\alpha$. And then, for a given i , it depends : if $\{t = \mu_\alpha^i\}$ is on the boundary of M_α^j , then we let

$$\psi_\alpha^j = \begin{cases} \frac{t + 2b_\alpha - \mu_\alpha^i}{b_\alpha} & \text{for } \mu_\alpha^i - 2b_\alpha \leq t \leq \mu_\alpha^i - b_\alpha \\ 1 & \text{for } \mu_\alpha^i - b_\alpha \leq t \leq \mu_\alpha^i \end{cases}$$

Otherwise, we let $\psi_\alpha^j = 0$ for $\mu_\alpha^i - 2b_\alpha \leq t \leq \mu_\alpha^i$. We proceed in the same way to define ψ_α^j on the other side of the hyperbolic cylinder P_α^i .

We clearly have that

$$\int_M |\nabla \psi_\alpha^j|_{g_\alpha}^2 dv_{g_\alpha} = \int_M |\nabla \psi_\alpha^j|_{h_\alpha}^2 dv_{h_\alpha} = \frac{2\pi m_j}{b_\alpha} \quad (3.59)$$

and that

$$\int_M (\psi_\alpha^j)^2 dv_{g_\alpha} \geq \text{Vol}_{g_\alpha}(M_\alpha^j(b_\alpha)) \quad (3.60)$$

for $1 \leq j \leq r$ where m_j is the number of connected components of ∂M_α^j . Note that $m_j \leq 2(3\gamma - 3)$.

We aim at testing these functions in the min-max formula for the first eigenvalue (see [104], page 88).

$$\lambda_\alpha = \inf_E \sup_{\varphi \in E \setminus \{0\}} \frac{\int_M |\nabla \varphi|_{g_\alpha}^2 dv_{g_\alpha}}{\int_M \varphi^2 dv_{g_\alpha}},$$

where the infimum is taken over the set of two-dimensional subspaces of $H^1(M)$. Then, for any two smooth functions φ and ψ on M with disjoint compact supports, we have that

$$\lambda_\alpha \leq \max \left\{ \frac{\int_M |\nabla \varphi|_{g_\alpha}^2 dv_{g_\alpha}}{\int_M \varphi^2 dv_{g_\alpha}}; \frac{\int_M |\nabla \psi|_{g_\alpha}^2 dv_{g_\alpha}}{\int_M \psi^2 dv_{g_\alpha}} \right\}. \quad (3.61)$$

Applying this to any pair of the above test functions, which all have disjoint compact supports, we get thanks to (3.57), (3.58), (3.59) and (3.60) that

$$\min \left\{ Vol_{g_\alpha} \left(P_\alpha^i(3b_\alpha) \right); Vol_{g_\alpha} \left(P_\alpha^j(3b_\alpha) \right) \right\} \leq \frac{C}{b_\alpha} \text{ for } i \neq j \in \{1, \dots, s\} \quad (3.62)$$

$$\min \left\{ Vol_{g_\alpha} \left(M_\alpha^i(b_\alpha) \right); Vol_{g_\alpha} \left(M_\alpha^j(b_\alpha) \right) \right\} \leq \frac{C}{b_\alpha} \text{ for } i \neq j \in \{1, \dots, r\} \quad (3.63)$$

$$\min \left\{ Vol_{g_\alpha} \left(P_\alpha^i(3b_\alpha) \right); Vol_{g_\alpha} \left(M_\alpha^j(b_\alpha) \right) \right\} \leq \frac{C}{b_\alpha} \text{ for } 1 \leq i \leq s \text{ and } 1 \leq j \leq r \quad (3.64)$$

where $C > 0$ is some fixed constant independent of the sequence (b_α) .

Let $D > 0$ that we shall fix later and let us assume that the conclusion of the claim does not hold. Let (a_α) be a sequence of positive real numbers with $a_\alpha \rightarrow +\infty$ and $\frac{a_\alpha}{\mu_\alpha^i} \rightarrow 0$ as $\alpha \rightarrow +\infty$ for all $1 \leq i \leq s$. Assume by contradiction that for any $i \in \{1, \dots, s\}$,

$$Vol_{g_\alpha} \left(P_\alpha^i(a_\alpha) \right) < 1 - \frac{D}{a_\alpha} \quad (3.65)$$

and that, for any $j \in \{1, \dots, r\}$,

$$Vol_{g_\alpha} \left(M_\alpha^j(9a_\alpha) \right) < 1 - \frac{D}{a_\alpha}. \quad (3.66)$$

Let $i \in \{1, \dots, s\}$. Assume that

$$Vol_{g_\alpha} \left(P_\alpha^i(3a_\alpha) \right) \geq \frac{10C}{3a_\alpha}. \quad (3.67)$$

Noting that $P_\alpha^i(3a_\alpha) \subset P_\alpha^j(a_\alpha)$, using (3.62) with $3b_\alpha = a_\alpha$, we get that

$$Vol_{g_\alpha} \left(P_\alpha^j(a_\alpha) \right) \leq \frac{3C}{a_\alpha} \text{ for } j \neq i. \quad (3.68)$$

Using (3.64) with $b_\alpha = a_\alpha$, we also get that

$$Vol_{g_\alpha} \left(M_\alpha^j(a_\alpha) \right) \leq \frac{C}{a_\alpha} \text{ for } 1 \leq j \leq r. \quad (3.69)$$

Since $\text{Vol}_{g_\alpha}(M) = 1$, we deduce from (3.68) and (3.69) that

$$\text{Vol}_{g_\alpha}(P_\alpha^i(a_\alpha)) \geq 1 - \frac{C(r+3s-3)}{a_\alpha}.$$

If we choose $D > C(r+3s-3)$, this contradicts (3.65) and thus proves that (3.67) can not hold. Thus, up to choose D large enough, we have proved that

$$\text{Vol}_{g_\alpha}(P_\alpha^i(3a_\alpha)) \leq \frac{10C}{3a_\alpha} \text{ for } 1 \leq i \leq s. \quad (3.70)$$

Let now $j \in \{1, \dots, r\}$ and assume that

$$\text{Vol}_{g_\alpha}(M_\alpha^j(3a_\alpha)) \geq \frac{2C}{3a_\alpha}. \quad (3.71)$$

Since $M_\alpha^j(3a_\alpha) \subset M_\alpha^j(9a_\alpha)$, we can use (3.63) with $b_\alpha = 9a_\alpha$ to write that

$$\text{Vol}_{g_\alpha}(M_\alpha^k(9a_\alpha)) \leq \frac{C}{9a_\alpha} \text{ for } k \neq j. \quad (3.72)$$

Using (3.64) with $b_\alpha = 3a_\alpha$, we can also write that

$$\text{Vol}_{g_\alpha}(P_\alpha^i(9a_\alpha)) \leq \frac{C}{3a_\alpha} \text{ for } 1 \leq i \leq s. \quad (3.73)$$

Combining (3.72) and (3.73) to the fact that $\text{Vol}_{g_\alpha}(M) = 1$, we deduce that

$$\text{Vol}_{g_\alpha}(M_\alpha^j(9a_\alpha)) \geq 1 - \frac{C(3s+r-1)}{9a_\alpha}.$$

Up to choose $D > \frac{C(3s+r-1)}{9}$, this contradicts (3.66) and thus proves that (3.71) can not hold. So we have proved that, up to choose D large enough,

$$\text{Vol}_{g_\alpha}(M_\alpha^j(3a_\alpha)) \leq \frac{2C}{3a_\alpha} \text{ for } 1 \leq j \leq r. \quad (3.74)$$

Now equations (3.70) and (3.74) together with the fact that $\text{Vol}_{g_\alpha}(M) = 1$ and that $a_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$ lead to a contradiction. Thus we have proved that (3.65) and (3.66) can not hold together, up to fix D large enough. This clearly permits to end the proof of the claim. \diamond

We shall now prove successively that both situations in claim 15 lead to a contradiction.

Claim 16. *If (a) holds in claim 15, then $\Lambda_1(\gamma) \leq 8\pi$.*

Proof. We follow ideas of Girouard [45]. Let $a_\alpha \rightarrow +\infty$ with $\frac{a_\alpha}{\mu_\alpha^i} \rightarrow 0$ as $\alpha \rightarrow +\infty$ for all $1 \leq i \leq s$. If (a) holds, there exists $1 \leq i \leq s$ such that

$$\text{Vol}_{g_\alpha}(P_\alpha^i(a_\alpha)) \geq 1 - \frac{D}{a_\alpha}. \quad (3.75)$$

Thus all the volume of g_α concentrates in the hyperbolic cylinder P_α^i . We shall omit the subscript i in the following and we shall identify P_α with \mathcal{C}_α , a subset of $S^1 \times \mathbb{R}$. We let $0 \leq \eta_\alpha \leq 1$

be a smooth cut-off function defined on M such that $\eta_\alpha \equiv 1$ on $P_\alpha^i(a_\alpha)$ and $\eta_\alpha \equiv 0$ on $M \setminus P_\alpha$. Moreover, we may choose it in such a way that

$$\int_M |\nabla \eta_\alpha|_{g_\alpha}^2 dv_{g_\alpha} \rightarrow 0 \text{ as } \alpha \rightarrow +\infty$$

thanks to the fact that $a_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We let $\Phi : P_\alpha \mapsto S^2$ be defined by

$$\Phi(t, \theta) = \frac{1}{1 + e^{2t}} (2e^t \cos \theta, 2e^t \sin \theta, e^{2t} - 1) .$$

This map Φ is conformal. Thanks to Hersch ([54], lemma 1.1), there exists a conformal diffeomorphism θ_α of S^2 such that

$$\int_{P_\alpha} (x \circ \theta_\alpha \circ \Phi) \eta_\alpha dv_{g_\alpha} = 0 . \quad (3.76)$$

We let $i_\alpha \in \{1, 2, 3\}$ be such that

$$\int_{P_\alpha(a_\alpha)} (x_{i_\alpha} \circ \theta_\alpha \circ \Phi)^2 \eta_\alpha^2 dv_{g_\alpha} \geq \frac{1}{3} \left(1 - \frac{D}{a_\alpha}\right) \quad (3.77)$$

and we set

$$u_\alpha = \eta_\alpha (x_{i_\alpha} \circ \theta_\alpha \circ \Phi) . \quad (3.78)$$

Such a i_α does obviously exist thanks to (3.75). It is then easily checked that

$$\int_M |\nabla u_\alpha|_{g_\alpha}^2 dv_{g_\alpha} \leq \frac{8\pi}{3} + o(1) .$$

Then we have that

$$\lambda_\alpha \leq \frac{\int_M |\nabla u_\alpha|_{g_\alpha}^2 dv_{g_\alpha}}{\int_M u_\alpha^2 dv_{g_\alpha}} \leq 8\pi + o(1) .$$

This ends the proof of the claim. \diamond

Claim 17. *If (b) holds in claim 15, then $\Lambda_1(\gamma) \leq \Lambda_1(\gamma - 1)$.*

Proof. If (b) holds, there exists $1 \leq j \leq r$ such that

$$\text{Vol}_{g_\alpha} \left(M_\alpha^j(9a_\alpha) \right) \geq 1 - \frac{D}{a_\alpha} . \quad (3.79)$$

Thus all the volume of g_α concentrates in the connected component M_α^j . We denote by \tilde{M}_α the connected component of $M \setminus (\gamma_\alpha^1 \cup \dots \cup \gamma_\alpha^s)$ which contains M_α^j . Then there exists a diffeomorphism $\tau_\alpha : \Sigma \mapsto \tilde{M}_\alpha$ with $\tau_\alpha^* h_\alpha = \bar{h}_\alpha$ where (Σ, \bar{h}_α) is a hyperbolic surface (non-compact). We have that

$$\bar{h}_\alpha \rightarrow h \text{ in } C_{loc}^\infty(\Sigma) \text{ as } \alpha \rightarrow +\infty .$$

We let for $\delta > 0$

$$\Sigma_\delta = \{x \in \Sigma \text{ s.t. } i_h(x) \geq \delta\}$$

so that

$$h_\alpha \rightarrow h \text{ in } C^\infty(\Sigma_\delta) \text{ as } \alpha \rightarrow +\infty .$$

Up to a subsequence, there exist a decreasing sequence $\delta_\alpha \rightarrow 0$ and an increasing sequence $a_\alpha \rightarrow +\infty$ such that

$$M_\alpha^j(9a_\alpha) \subset \tau_\alpha(\Sigma_{\delta_\alpha}) . \quad (3.80)$$

We let $c = [h]$. We denote by $(\hat{\Sigma}, \hat{c})$ the compactification of the cusps of (Σ, c) (see Hummel [55] sections I.5, IV.2, IV.5 and V.1) : $(\hat{\Sigma} \setminus \{p_1, \dots, p_t\}, \hat{c})$ is conformal to (Σ, c) . Note that $\hat{\Sigma}$ has genus less than or equal to $\gamma - 1$.

We also set

$$\bar{\Phi}_\alpha = \Phi_\alpha \circ \tau_\alpha$$

and

$$\bar{g}_\alpha = \tau_\alpha^* g_\alpha .$$

We shall study the asymptotic behaviour of the harmonic map $\bar{\Phi}_\alpha : (\Sigma, \bar{h}_\alpha) \rightarrow \mathbb{S}^k$. By theorem 2.2 of Zhu [115], there exist $x_1, \dots, x_N \in \Sigma$ and a harmonic map $\Phi : (\Sigma, h) \mapsto \mathbb{S}^k$ such that

$$\bar{\Phi}_\alpha \rightarrow \Phi \text{ in } C_{loc}^\infty(\Sigma \setminus \{x_1, \dots, x_N\}) \text{ as } \alpha \rightarrow +\infty$$

and

$$\int_{\tau_\alpha(\Sigma_{\delta_\alpha})} |\nabla \Phi_\alpha|_{h_\alpha}^2 dv_{h_\alpha} \rightarrow \int_\Sigma |\nabla \Phi|_h^2 dv_h + \sum_{i=1}^N E_i$$

where the E_i 's correspond to the energies lost at the blow up points x_i . Since λ_α is uniformly bounded from below by $8\pi + \epsilon_0$ (because $\Lambda_1(\gamma) > 8\pi$, see theorem 4), we can adapt claim 5 to prove that all the E_i 's are 0. Now, thanks to theorem 3.6 of Sacks-Uhlenbeck [100], the harmonic map Φ can be extended to $\hat{\Sigma}$ by

$$\hat{\Phi} : (\hat{\Sigma}, \hat{c}) \mapsto \mathbb{S}^k .$$

Choosing $g_0 \in \hat{c}$ a regular metric, we have by conformal invariance of the L^2 -norm of the gradient, (3.80), (3.79) and what we just said that

$$\int_{\hat{\Sigma}} |\nabla \hat{\Phi}|_{g_0}^2 dv_{g_0} = \Lambda_1(\gamma) .$$

We let

$$g = \frac{|\nabla \hat{\Phi}|_{g_0}^2}{\Lambda_1(\gamma)} g_0$$

so that $\text{Vol}_g(\hat{\Sigma}) = 1$. Let $\psi \in C^\infty(\hat{\Sigma})$ be a first eigenfunction of g . Let also

$$\rho_\epsilon \in C_c^\infty(\hat{\Sigma} \setminus \{p_1, \dots, p_t\})$$

be such that

$$\rho_\epsilon = 1 \text{ on } \hat{\Sigma} \setminus \bigcup_{i=1}^t B_{p_i}(\epsilon)$$

and such that

$$\int_{\hat{\Sigma}} |\nabla \rho_\epsilon|_g^2 dv_g \rightarrow 0 \text{ as } \epsilon \rightarrow 0 .$$

Then we write that

$$\lambda_\alpha \leq \frac{\int_\Sigma |\nabla(\rho_\epsilon \psi)|_{\bar{g}_\alpha}^2 dv_{\bar{g}_\alpha}}{\int_\Sigma (\rho_\epsilon \psi)^2 dv_{\bar{g}_\alpha} - \left(\int_\Sigma \rho_\epsilon \psi dv_{\bar{g}_\alpha}\right)^2} .$$

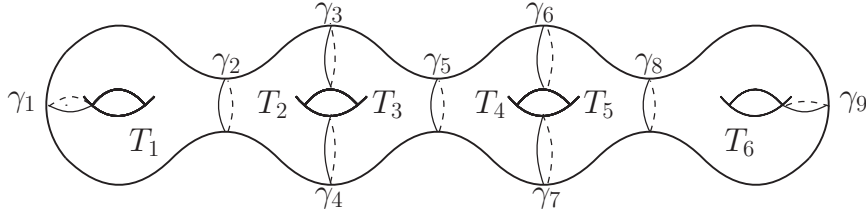


FIGURE 3.1 – Construction in genus $\gamma = 4$

Passing to the limit as $\alpha \rightarrow +\infty$, we get that

$$\Lambda_1(\gamma) \leq \frac{\int_{\Sigma} |\nabla(\rho_{\epsilon}\psi)|_g^2 dv_g}{\int_{\Sigma} (\rho_{\epsilon}\psi)^2 dv_g - \left(\int_{\Sigma} \rho_{\epsilon}\psi dv_g\right)^2}.$$

Passing to the limit as $\epsilon \rightarrow 0$, it is easily checked that this leads to

$$\Lambda_1(\gamma) \leq \lambda_1(g) \leq \Lambda_1(\gamma - 1).$$

This ends the proof of the claim. \diamond

Thus we have proved that, if $\Lambda_1(\gamma) > \Lambda_1(\gamma - 1)$, then $\Lambda_1(\gamma)$ is achieved by a smooth metric, up to a finite set of conical singularities. This ends the proof of theorem 5.

3.5 The infimum of the first conformal eigenvalue over all conformal classes

In this section, we prove theorem 6. Fix $\gamma \geq 2$ since the result is already known in genera 0 ([54]) and 1 ([45]). We consider a sequence M^n of hyperbolic surfaces (with metric h_n) obtained by gluing $2\gamma - 2$ pairs of pants T_n^j : these are surfaces containing $3\gamma - 3$ closed geodesics $\gamma_n^1, \dots, \gamma_n^{3\gamma-3}$ of length $\epsilon_n \rightarrow 0$ as $n \rightarrow +\infty$ (see figure 3.1).

Each geodesic γ_n^i has a neighbourhood P_i^n isometric to the truncated cylinder $C_n = (-v_n, v_n) \times (0, 2\pi)$ with

$$v_n = \frac{\pi^2}{\epsilon_n} - \frac{2\pi}{\epsilon_n} \arctan\left(\sinh \frac{\epsilon_n}{2}\right)$$

endowed with the conformally flat metric

$$g_n = \left(\frac{\epsilon_n}{2\pi \cos(\frac{\epsilon_n}{2\pi}t)}\right)^2 (dt^2 + d\theta^2).$$

We choose that the negative part of P_i^n , that is $t \leq 0$, is in T_k while the positive part is in T_{k+1} .

We let $a_n \rightarrow +\infty$ with $\frac{a_n}{v_n} \rightarrow 0$ as $n \rightarrow +\infty$. We let ψ , φ_l , φ_r , θ_l , θ_r be defined on C_n as

follows (depending on n but we drop the subscript n) :

$$\begin{aligned}\psi &= \begin{cases} 0 & \text{for } -v_n < t \leq -v_n + a_n \\ \frac{t + v_n - a_n}{a_n} & \text{for } -v_n + a_n < t \leq -v_n + 2a_n \\ 1 & \text{for } -v_n + 2a_n < t < v_n - 2a_n \\ \frac{v_n - a_n - t}{a_n} & \text{for } v_n - 2a_n \leq t < v_n - a_n \\ 0 & \text{for } v_n - a_n \leq t < v_n \end{cases} \\ \varphi_l &= \begin{cases} 1 & \text{for } -v_n < t \leq -v_n + a_n \\ \frac{2a_n - v_n - t}{a_n} & \text{for } -v_n + a_n \leq t < -v_n + 2a_n \\ 0 & \text{for } -v_n + 2a_n \leq t < v_n \end{cases} \\ \theta_l &= \begin{cases} \frac{t + v_n}{a_n} & \text{for } -v_n \leq t \leq -v_n + a_n \\ 1 & \text{for } -v_n + a_n \leq t \leq -v_n + 3a_n \\ \frac{4a_n - v_n - t}{a_n} & \text{for } -v_n + 3a_n \leq t \leq -v_n + 4a_n \\ 0 & \text{for } -v_n + 4a_n \leq t \leq v_n \end{cases}\end{aligned}$$

and $\theta_r(t) = \theta_l(-t)$, $\psi_r(t) = \psi_l(-t)$.

We can now define the following test functions on M^n : for $1 \leq i \leq 3\gamma - 3$,

$$\psi_i = \begin{cases} \psi & \text{on } P_i \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\begin{aligned}\theta_{l,i} &= \begin{cases} \theta_l & \text{on } P_i \\ 0 & \text{elsewhere} \end{cases} \\ \theta_{r,i} &= \begin{cases} \theta_r & \text{on } P_i \\ 0 & \text{elsewhere} \end{cases}\end{aligned}$$

For $1 \leq k \leq \gamma - 2$,

$$\varphi_{2k+1} = \begin{cases} \varphi_r & \text{on } P_{3k} \\ \varphi_r & \text{on } P_{3k+1} \\ \varphi_l & \text{on } P_{3k+2} \\ 1 & \text{elsewhere in } T_{2k+1} \\ 0 & \text{elsewhere in } M^n \end{cases}$$

and

$$\varphi_{2k} = \begin{cases} \varphi_r & \text{on } P_{3k-1} \\ \varphi_l & \text{on } P_{3k} \\ \varphi_l & \text{on } P_{3k+1} \\ 1 & \text{elsewhere in } T_{2k} \\ 0 & \text{elsewhere in } M^n \end{cases}$$

We also define

$$\varphi_1 = \begin{cases} \varphi_r + \varphi_l & \text{on } P_1 \\ \varphi_l & \text{on } P_2 \\ 1 & \text{elsewhere in } T_1 \\ 0 & \text{elsewhere in } M^n \end{cases}$$

and

$$\varphi_{2\gamma-2} = \begin{cases} \varphi_r + \varphi_l & \text{on } P_{3\gamma-3} \\ \varphi_r & \text{on } P_{3\gamma-2} \\ 1 & \text{elsewhere in } T_{2\gamma-2} \\ 0 & \text{elsewhere in } M^n \end{cases}$$

We let now $g_n \in [h_n]$ with volume 1 be such that

$$\lambda_n = \lambda_1(g_n) = \sup_{g \in [h_n]} \lambda_1(g) \text{Vol}_g(M^n) .$$

Such a g_n does exist thanks to theorem 4.

We denote by \mathcal{E} the set of all the above functions defined on M^n . Note that all these functions have an L^2 -norm (with respect to g_n) of their gradient converging to 0 as $n \rightarrow +\infty$ (using the conformal invariance of this norm). Then, with the characterization (3.61) of the

first eigenvalue, if u and v are two functions in \mathcal{E} with disjoint compact supports, we have that

$$\lambda_n \min \left\{ \int_{M^n} u^2 dv_{g_n}; \int_{M^n} v^2 dv_{g_n} \right\} \leq o(1) \text{ as } n \rightarrow \infty. \quad (3.81)$$

Thanks to this remark, we will prove that one of the following situations must occur :

a) Up to a subsequence, there exists $1 \leq i \leq 3\gamma - 3$ such that

$$\int_{M^n} \tau_i^2 dv_{g_n} \rightarrow 1 \text{ as } n \rightarrow +\infty$$

where

$$\tau_i = \max \{ \theta_{l,i}, \theta_{r,i}, \psi_i \}.$$

b) Up to a subsequence, there exists $1 \leq j \leq 2\gamma - 2$ such that

$$\int_{M^n} \eta_j^2 dv_{g_n} \rightarrow 1 \text{ as } n \rightarrow +\infty$$

where for $1 \leq k \leq \gamma - 2$ we define :

$$\eta_{2k+1} = \max \{ \varphi_{2k+1}; \theta_{r,3k}; \theta_{r,3k+1}; \theta_{l,3k+2} \}$$

$$\eta_{2k} = \max \{ \varphi_{2k}; \theta_{r,3k-1}; \theta_{r,3k}; \theta_{l,3k+1} \}$$

$$\eta_1 = \max \{ \varphi_1; \theta_{l,1}; \theta_{r,1}; \theta_{l,2} \}$$

$$\eta_{2\gamma-2} = \max \{ \varphi_{2\gamma-2}; \theta_{r,3\gamma-2}; \theta_{l,3\gamma-3}; \theta_{r,3\gamma-3} \}.$$

Indeed, we set

$$\mathcal{F} = \{ u \in \mathcal{E}; \int_M u^2 \not\rightarrow 0 \text{ as } n \rightarrow +\infty \}.$$

Since we have

$$\int_{M^n} \left(\max_{u \in \mathcal{F}} \{ u^2 \} + \max_{u \in \mathcal{E} \setminus \mathcal{F}} \{ v^2 \} \right) dv_{g_n} \geq \int_{M^n} \max_{u \in \mathcal{E}} \{ u^2 \} dv_{g_n} = 1$$

we easily get that

$$\int_{M^n} \left(\max_{u \in \mathcal{F}} u \right)^2 dv_{g_n} \geq 1 - \sum_{v \in \mathcal{E} \setminus \mathcal{F}} \int_{M^n} v^2 dv_{g_n} \rightarrow 1 \text{ as } n \rightarrow +\infty. \quad (3.82)$$

Then $\mathcal{F} \neq \emptyset$ and we distinguish two cases :

(i) There exists $1 \leq j \leq 2\gamma - 2$ such that $\varphi_j \in \mathcal{F}$. Then, up to a subsequence, $\int_{M^n} \varphi_j^2 dv_{g_n}$ is uniformly bounded below and thanks to (3.81), we get that \mathcal{F} contains at most two functions, with non-disjoint supports. Taking the maximum of these two functions, we easily obtain b) from (3.82).

(ii) For all $1 \leq j \leq 2\gamma - 2$, $\varphi_j \in \mathcal{E} \setminus \mathcal{F}$. Since $\mathcal{F} \neq \emptyset$, there exists $1 \leq i \leq 3\gamma - 3$ such that

$$\{ \psi_i; \theta_{l,i}; \theta_{r,i} \} \cap \mathcal{F} \neq \emptyset.$$

Then up to a subsequence, $\int_{M^n} \tau_i^2 dv_{g_n}$ is uniformly bounded below and thanks to (3.81), we get that $\mathcal{F} \subset \{\psi_i; \theta_{l,i}; \theta_{r,i}\}$, and with (3.82), we obtain a).

In both cases a) and b), we are in the situation of the lemma below and we deduce from it that $\lambda_n \leq 8\pi + o(1)$. This concludes the proof of theorem 6. \diamond

It remains to prove the following lemma we used during the previous proof :

Lemma. *Let Σ be a compact orientable surface of genus 0 with a boundary of k connected components endowed with a sequence g_n of metrics. Assume that there exists a sequence of functions $\eta_n : \Sigma \mapsto \mathbb{R}$ in $H^1 \cap C^0$ such that :*

- i) $0 \leq \eta_n \leq 1$.
- ii) η_n is compactly supported in $\overset{\circ}{\Sigma}$.
- iii) $\int_{\Sigma} \eta_n^2 dv_{g_n} \rightarrow 1$ as $n \rightarrow +\infty$.
- iv) $\int_{\Sigma} |\nabla \eta_n|_{g_n}^2 dv_{g_n} \rightarrow 0$ as $n \rightarrow +\infty$.

Then there exists $u_n : \Sigma \mapsto \mathbb{R}$ in $H^1 \cap C^0$ compactly supported in $\overset{\circ}{\Sigma}$ such that $\int_{\Sigma} u_n dv_{g_n} = 0$ and

$$\frac{\int_{\Sigma} |\nabla u_n|_{g_n}^2 dv_{g_n}}{\int_{\Sigma} u_n^2 dv_{g_n}} \leq 8\pi + o(1) .$$

Proof. We first build a conformal diffeomorphism $\Psi_n : (\overset{\circ}{\Sigma}, g_n) \rightarrow (\Sigma_n, h)$ where $\Sigma_n = \Psi_n(\overset{\circ}{\Sigma}) \subset \mathbb{S}^2$ and h is the round metric of \mathbb{S}^2 .

Let U_1, \dots, U_k some disjoint neighbourhoods of each connected component of the boundary which are diffeomorphic to annulus and such that, by the uniformization theorem for annuli (see [55], I.5), we get some conformal diffeomorphisms

$$\Phi_n^i : (U_i, g_n) \rightarrow (A_{r_n^i}, \xi)$$

where $0 < r_n^i < 1$ and for $0 < r < 1$, $A_r \subset \mathbb{D}$ is the annulus

$$A_r = \{z \in \mathbb{C}; r < |z| < 1\} .$$

Gluing k copies of \mathbb{D} instead of $A_{r_n^i}$, one can define a natural surface $\widetilde{\Sigma}_n$ endowed with a conformal structure $[\widetilde{g}_n]$ which extends $(\overset{\circ}{\Sigma}, [g_n])$. $\widetilde{\Sigma}_n$ has a zero genus and by the uniformization theorem, there is a conformal diffeomorphism

$$\widetilde{\Psi}_n : (\widetilde{\Sigma}_n, [\widetilde{g}_n]) \rightarrow (\mathbb{S}^2, [h]) .$$

Setting $\Psi_n = \widetilde{\Psi}_n|_{\overset{\circ}{\Sigma}}$ gives the expected conformal map. Following the arguments of claim 16 permits to end the proof. \diamond

Chapitre 4

Maximiser les valeurs propres de Laplace sur une surface

Dans ce chapitre, nous étudions les fonctionnelles de valeurs propres de Laplace $\lambda_k(\Sigma, g) Vol_g(\Sigma)$ sur des surfaces compactes sans bord. Nous démontrons que sous certaines hypothèses naturelles, ces fonctionnelles ont une métrique maximale. Ceci donne aussi l'existence d'immersions minimales dans des sphères par des k -èmes fonctions propres.

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4.1 Introduction

Let Σ be a smooth compact connected surface without boundary. Given a Riemannian metric g on Σ , the spectrum of $\Delta_g = -\operatorname{div}_g \nabla$, the Laplace-Beltrami operator, is a sequence

$$0 = \lambda_0 < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \cdots \leq \lambda_k(\Sigma, g) \leq \cdots$$

of eigenvalues counted with multiplicity. We can view all these eigenvalues as functionals of the metric g . For obvious scaling reasons, it is more interesting to look at the functionals $\lambda_k(\Sigma, g) \operatorname{Vol}_g(\Sigma)$. Then one can try to get bounds on these eigenvalues, depending on the geometry of g or depending only on the topology (genus and orientability) of the surface. One can also try to find critical points of these functionals (under possibly additional constraints). Of course, these functionals are not C^1 so we have to deal with critical points in a generalized sense. But these critical points are really interesting to search for because they come with a corresponding minimal immersion of the surface into some sphere.

The study of eigenvalues of the Laplacian on surfaces has a very long history but one breakthrough was obtained by Yang and Yau [113] in 1980. They proved that, in dimension 2, the first eigenvalue can not be made arbitrarily large (except by reducing the volume of the

surface). More precisely they proved that, on any smooth compact orientable surface of genus γ , for any metric g on Σ ,

$$\lambda_1(\Sigma, g) \text{Vol}_g(\Sigma) \leq 8\pi \left\lceil \frac{\gamma+3}{2} \right\rceil .$$

In the case of the sphere ($\gamma = 0$), they recovered the celebrated isoperimetric inequality of Hersch [54] who proved in 1970 that the first eigenvalue on any sphere of volume 1 was less than or equal to 8π , with equality for and only for the round sphere. Later, this result was extended by Li and Yau [74] to non-orientable surfaces and a link between various natural functionals on surfaces was made. They showed in particular that

$$\lambda_1(\Sigma, g) \text{Vol}_g(\Sigma) \leq 2V_c(\Sigma) \leq 2 \int_{\Sigma} |H|^2 d\sigma$$

for any metric g on Σ , for any immersion of Σ into some \mathbb{R}^n with mean curvature H , where $V_c(\Sigma)$ is the conformal volume of Σ , defined as the minimum over branched immersions of Σ into some sphere S^n of the maximum of the volume of this branched immersion composed with a conformal automorphism of the sphere. These functionals are all really interesting to study, their critical points leading to various interesting surfaces. The Willmore functional has been the subject of a lot of recent works (see for instance the surveys Marques-Neves [76] and Rivière [98]). For the conformal volume, much less studied, we refer to the original Li-Yau [74], to Montiel-Ros [80] or to the more recent Rivière [99].

Then Korevaar [67] extended this result of Li and Yau to higher eigenvalues by proving that there exists a universal constant such that, on any smooth compact orientable surface of genus γ , for any $k \geq 1$ and for any metric g on Σ ,

$$\lambda_k(g) \text{Vol}_g(\Sigma) \leq Ck(\gamma+1) .$$

Thus a natural way to find critical points of these functionals is to maximize them. If Σ is an orientable surface of genus γ , we define the topological invariant

$$\Lambda_k(\gamma) = \sup_g \lambda_k(\Sigma, g) \text{Vol}_g(\Sigma) .$$

And the aim is to prove that it is achieved by some smooth enough metric. This variational problem is deeply related to minimal surfaces theory. A maximal metric for $\Lambda_k(\gamma)$ is the pull-back of the induced metric of a minimal immersion of Σ into some sphere S^n , possibly with branched points, and the coordinate functions of this immersion (as a map into \mathbb{R}^{n+1}) are k -th eigenfunctions associated to $\Lambda_k(\gamma)$ (as proved in Fraser-Schoen [39] for instance). Conversely, since Takahashi [108], we know that the coordinates of any minimal immersion into a sphere are necessarily eigenfunctions associated to the same eigenvalue λ . Moreover, the pull-back of the induced metric by this minimal immersion is critical (in the sense of [34] and [83]) for the λ_k functional where k is an integer such that $\lambda_k = \lambda$. Classical existence results of minimal immersions of surfaces into the sphere S^n were given by Lawson [72] for $n = 3$ and Bryant [11] for $n = 4$. Since then, despite numerous works on the subject, a complete classification of minimal surfaces into spheres is very far from being discovered, even for embedded ones. For instance, Yau [114] conjectured that minimal embeddings in S^3 all come from first eigenfunctions. Recently, Brendle [7] proved the Lawson conjecture which states that the only minimal embedded 2-torus in S^3 is the Clifford torus, (which is known

to be the only minimal torus immersed into S^3 by first eigenfunctions since Montiel and Ros [80]). Looking for critical metrics for λ_k with fixed k is another point of view for the study of minimal immersions into spheres. This is the subject of our first theorem.

Before stating our theorem, let us mention that it can be proved by standard gluing techniques, see for instance Colbois-El Soufi [22], that

$$\Lambda_k(\gamma) \geq \max_{\substack{i_1+\dots+i_s=k \\ \forall m, i_m \geq 1 \\ \gamma_1+\dots+\gamma_s \leq \gamma \\ \gamma_1 < \gamma \text{ if } s=1}} \Lambda_{i_1}(\gamma_1) + \dots + \Lambda_{i_s}(\gamma_s) . \quad (4.1)$$

Our existence result reads then as follows :

Theorem 8. *Let Σ be a compact orientable surface without boundary of genus γ . If the inequality (4.1) is strict, then there exists a metric g on Σ which is smooth except maybe at a finite set of conical singularities such that $\Lambda_k(\gamma) = \lambda_k(\Sigma, g) \text{Vol}_g(\Sigma)$. Moreover this metric g is the pull-back of a minimal immersion from Σ into some sphere S^n by k -th eigenfunctions. At last, under this condition, the set of such maximal metrics is compact.*

In the case $k = 1$, this theorem has already been proved in Petrides [91]. When $k = 1$, the gap assumption of the theorem reads as $\Lambda_1(\gamma) > \Lambda_1(\gamma - 1)$. This condition holds for $\gamma = 1$, see [83], there are some numerical evidences that it holds for $\gamma = 2$, see [56], and it does hold for an infinite number of $\gamma \geq 1$ since, combining a remark of Fraser and Schoen [39] based on results by Buser, Burger and Dodziuk [14] and Brooks and Makover [10] with (4.1), we know that

$$\Lambda_k(\gamma) \geq \frac{3\pi}{4}(\gamma - 1) + 8\pi(k - 1) \quad (4.2)$$

for γ large enough.

Note that the gap assumption asserting that (4.1) is strict is somewhat necessary since, for instance, it has been proved in Nadirashvili [84] and Petrides [92] that $\Lambda_2(0) = 2\Lambda_1(0) = 16\pi$ and the maximum is not achieved. Note also that the fact that the maximal metric obtained is not completely smooth and may have conical singularities is the optimal regularity result one can hope since the minimal immersion we obtain may have branched points. Moreover, for $k = 1$ and $\gamma = 2$, it is conjectured that there are maximal metrics which have conical singularities (see [56]).

Our proof relies on a second theorem and on the simple remark that maximizing the k -th eigenvalue among metrics with fixed volume can be done first by maximizing it among metrics in a fixed conformal class and then maximizing among conformal classes. That's why we introduce the conformal invariant

$$\Lambda_k(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \lambda_k(\Sigma, \tilde{g}) \text{Vol}_{\tilde{g}}(\Sigma)$$

on any smooth closed surface Σ equipped with a metric g . Here $[g]$ denotes the conformal class of g consisting of all metrics which are a multiple of g by a smooth positive function. Then we have that

$$\Lambda_k(\gamma) = \sup_{[g]} \Lambda_k(\Sigma, [g]) .$$

The starting point of the proof is the following remark : it is more convenient (even if not simple) to maximize our functional in a given conformal class because everything depends only on a function. Then, given the existence of this maximizing metric in a given conformal class, we can pick up a special maximizing sequence for $\Lambda_k(\gamma)$ which consists in maximizers in their own conformal classes. These maximizers come, as we will see, with corresponding harmonic maps into spheres and the proof of Theorem 8 relies on a careful asymptotic analysis of these harmonic maps when the conformal class degenerates. With the assumption that (4.1) is strict, we can in fact rule out this situation and prove a convergence result on the sequence of conformal classes.

Once again, one can check, by gluing techniques, see [21], that

$$\Lambda_k(\Sigma, [g]) \geq \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} \left(\Lambda_{k-j}(\Sigma, [g]) + \sum_{m=1}^s \Lambda_{i_m}(\mathbb{S}^2, [can]) \right) \quad (4.3)$$

for all $(\Sigma, [g])$ and $k \geq 1$. It is believed that $\Lambda_k(\mathbb{S}^2, [can]) = 8\pi k$ for all $k \geq 1$ and this has been proved for $k = 1$ ([54]) and for $k = 2$ ([84] and [92]). If this is true, then (4.3) reduces to

$$\Lambda_k(\Sigma, [g]) \geq \Lambda_{k-1}(\Sigma, [g]) + 8\pi.$$

Concerning the maximization of this conformal invariant, we prove :

Theorem 9. *Let (Σ, g) be a closed Riemannian surface and let $k \geq 1$. If (4.3) is strict, then there exists a maximal metric $\tilde{g} \in [g]$, smooth except maybe at a finite number of conical singularities, such that $\Lambda_k(\Sigma, [g]) = \lambda_k(\Sigma, \tilde{g}) \text{Vol}_{\tilde{g}}(\Sigma)$. Moreover, there exists a family of orthogonal k -th eigenfunctions for \tilde{g} giving rise to a smooth harmonic function into some sphere \mathbb{S}^n . At last, we have that the set of such smooth maximal metrics is compact.*

In the case $k = 1$, the assumption reads as $\Lambda_1(\Sigma, [g]) > \Lambda_1(\mathbb{S}^2, [can]) = 8\pi$ which holds true as soon as Σ is not diffeomorphic to \mathbb{S}^2 as proved in Petrides [91]. It should also hold true for $k = 2$, for some specific surfaces of genus 2. Indeed, thanks to (4.1) and to the values $\Lambda_1(1) = \frac{8\pi^2}{\sqrt{3}}$ and $\Lambda_1(2) = 16\pi$ respectively obtained in [83] and conjectured in [56], we get that $\Lambda_2(2) \geq 2\Lambda_1(1) = \frac{16\pi^2}{\sqrt{3}} > 24\pi = \Lambda_1(2) + \Lambda_1(0)$. This indicates in particular that there should be an open set of conformal classes $[g]$ which satisfy that (4.3) is strict on surfaces Σ of genus 2 for $k = 2$, that is $\Lambda_2(\Sigma, [g]) > \Lambda_1(\Sigma, [g]) + 8\pi$. Therefore, Theorem 9 applies in these cases and we get smooth (outside conical singularities) maximal metrics.

If Theorem 9 applies, maximal metrics for $\Lambda_k(\Sigma, [g])$ exist and the conformal factor related to g of such a maximal metric \tilde{g} is $|\nabla\Phi|_{\tilde{g}}^2$ where Φ is some smooth harmonic map from Σ into some sphere \mathbb{S}^n . Conical singularities naturally appear as zeros of $|\nabla\Phi|_{\tilde{g}}^2$ which are isolated as proved in Salamon [101].

Theorem 9 was already proved for $k = 1$ in Petrides [91] using that any maximizing sequences for $\Lambda_1(\Sigma, [g])$ is compact thanks to the assumption $\Lambda_1(\Sigma, [g]) > 8\pi$. Indeed, if some concentration points appear, some test functions, particular to λ_1 , permit to prove that $\Lambda_1(\Sigma, [g]) = 8\pi$ (see Kokarev [65], Lemma 3.1). A non-concentration assumption is sufficient to prove that the specific maximizing sequence selected by a regularization process converges

to a "smooth" metric. However, in the general case, we cannot remove a priori the concentration points of maximizing sequences, even with the assumption that (4.3) is strict. Thus, assuming that concentration points occur, we have to perform a multi-bubble asymptotic analysis on the specific maximizing sequence to obtain regularity estimates and convergence to a "smooth" metric at each scale of concentration. We also have to verify carefully that no energy is lost in the necks. Only at the end of the proof, we obtain natural test functions which permit to prove the case of equality in (4.3), and Theorem 9 follows.

Note that we do not prove either that any maximizing sequence does converge to a maximal metric nor that maximizers in a possible "weaker sense" are regular. Instead, as was initiated by Fraser and Schoen [41] for the study of the first Steklov eigenvalue, we carefully select a maximizing sequence by a regularization process which does converge to a smooth maximal metric.

The paper is organized as follows :

In Section 4.2, we introduce some notations and recall some more or less classical tools we shall use during the proof. Section 4.3 is devoted to the set up of the proof of Theorem 9, proof carried out in Sections 4.4 to 4.7. We refer to the end of Section 4.3 for a detailed sketch of the proof of Theorem 9.

We prove Theorem 8 in Section 7, studying a maximizing sequence of metrics for $\Lambda_k(\gamma)$ whose k -th eigenvalue is maximal in its conformal class (given by Theorem 9). The proof of the convergence of this sequence of metrics can be done thanks to the study of the asymptotics of the harmonic maps into some spheres S^n they define. In particular, thanks to the assumption that (4.1) is strict, we remove all the degenerations which could occur.

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4.2 Preliminaries

4.2.1 Notations

Let (M, g) be a smooth Riemannian surface with $\text{Vol}_g(M) = 1$.

We denote by $\lambda_k(M, g)$ the k -th eigenvalue of the Laplacian. It satisfies the min-max variational characterization :

$$\lambda_k(M, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_M \phi^2 dv_g},$$

where the infimum is taken over the spaces of smooth functions E_{k+1} of dimension $k + 1$.

We denote by $\mathcal{M}(M)$ the set of positive Radon measures provided with the weak* topology and $\mathcal{M}_1(M)$ the subset of probability measures. For an open set $\Omega \subset M$ we denote by $\lambda_*(\Omega, g)$ the first Dirichlet eigenvalue in (Ω, g) .

For all the paper, we fix $\delta > 0$, a constant $C_0 > 1$ and a family $(x_l)_{l=1, \dots, L}$ of points in M and smooth functions $v_l : M \mapsto \mathbb{R}$ such that

- for any $l \in \{1, \dots, L\}$, the metric $g_l = e^{-2v_l}g$ is a flat metric in the ball $B_{g_l}(x_l, 2\delta) = \Omega_l$.

- $M = \bigcup_{l=1}^L \omega_l$ where $\omega_l = B_{g_l}(x_l, \delta)$.
- For any $1 \leq l \leq L$, $C_0^{-2} \leq e^{2v_l} \leq C_0^2$.
- For any $x \in \omega_l$ and $0 < r < \delta$, $B_g(x, C_0^{-1}r) \subset B_{g_l}(x, r) \subset B_g(x, C_0 r)$

For $1 \leq l \leq L$ and a point $z \in \mathbb{D}_{2\delta}(0)$, we let

$$e^{2\tilde{v}_l(z)} = e^{2v_l(\exp_{g_l, x_l}(z))} \text{ and } \tilde{z}^l = \exp_{g_l, x_l}(z)$$

and for $x \in \Omega_l$ and a set $\Omega \subset \Omega_l$,

$$\tilde{x}^l = \exp_{g_l, x_l}^{-1}(x) \text{ and } \tilde{\Omega}^l = \exp_{g_l, x_l}^{-1}(\Omega) .$$

For a smooth density e^{2u} with $e^{2u}g \in [g]$, we let

$$e^{2\tilde{u}^l(z)} = e^{2\tilde{v}_l(z)} e^{2u(\exp_{g_l, x_l}(z))}$$

so that for $\Omega \subset \Omega_l$,

$$\int_{\Omega} e^{2u} dv_g = \int_{\tilde{\Omega}^l} e^{2\tilde{u}^l} dx .$$

For other functions $\phi \in L^1(M)$ or measures $\nu \in \mathcal{M}(M)$, we let

$$\tilde{\phi}^l(z) = \phi(\exp_{g_l, x_l}(z)) \text{ and } \tilde{\nu}^l = \exp_{g_l, x_l}^*(\nu) .$$

Let $p_{\epsilon}(x, y)$ be the heat kernel of (M, g) at time $\epsilon > 0$. Then, for $y, z \in \Omega_l$, we let

$$\tilde{p}_{\epsilon}^l(z, y) = e^{2\tilde{v}_l(z)} p_{\epsilon}(\exp_{g_l, x_l}(z), \exp_{g_l, x_l}(y))$$

so that for a density $e^{2u(x)} = \int_{\Omega} p_{\epsilon}(x, y) d\nu(y)$ for $\Omega \subset \Omega_l$ and some measure ν , we have

$$e^{2\tilde{u}^l(z)} = \int_{\tilde{\Omega}^l} \tilde{p}_{\epsilon}^l(z, y) d\tilde{\nu}(y)$$

and for $\phi \in L^1(M)$,

$$\int_{\tilde{\Omega}^l} \tilde{\phi}^l(z) \tilde{p}_{\epsilon}^l(z, \tilde{y}^l) dz = \int_{\Omega} \phi(x) p_{\epsilon}(x, y) dv_g(x) .$$

When the context is clear, we drop the exponent l in all the notations.

Now, for parameters $a \in \mathbb{R}^2$ and $\alpha > 0$, we denote the following rescaled objects by

$$\hat{x} = \frac{\tilde{x} - a}{\alpha} \text{ and } \hat{\Omega} = \frac{\tilde{\Omega} - a}{\alpha} ,$$

$$e^{2\hat{u}(z)} = \alpha^2 e^{2\tilde{u}(\alpha z + a)}, \hat{\phi}(z) = \tilde{\phi}(\alpha z + a) \text{ and } \hat{\nu} = H_{a, \alpha}^*(\tilde{\nu}) ,$$

$$\hat{p}_{\epsilon}(z, y) = \alpha^2 \tilde{p}_{\epsilon}^l(\alpha z + a, \alpha y + a) ,$$

where $H_{a, \alpha}(x) = \alpha x + a$, so that if $e^{2u(x)} = \int_{\Omega} p_{\epsilon}(x, y) d\nu(y)$, we have

$$e^{2\hat{u}(z)} = \int_{\hat{\Omega}} \hat{p}_{\epsilon}(z, y) d\hat{\nu}(y)$$

and

$$\int_{\hat{\Omega}} \phi(z) \hat{p}_{\epsilon}(z, \hat{y}) dz = \int_{\Omega} \phi(x) p_{\epsilon}(x, y) dv_g(y) .$$

We also let for $z \in \mathbb{R}^2$,

$$\check{z} = \exp_{g_l, x_l}(\alpha z + a)$$

so that $\hat{\check{z}} = z$ and

$$\check{\Omega} = \exp_{g_l, x_l}(\alpha \Omega + a) .$$

4.2.2 Uniform estimates on the heat kernel

The heat kernel $p_\epsilon(x, y)$ of a compact Riemannian surface (M, g) at time $\epsilon > 0$ satisfies the following uniform estimates as $\epsilon \rightarrow 0$

$$p_\epsilon(x, y) = \frac{e^{-\frac{d_g(x, y)^2}{4\epsilon}}}{4\pi\epsilon} (a_0(x, y) + \epsilon a_1(x, y) + \epsilon^2 a_2(x, y) + O(\epsilon^3)) \text{ as } \epsilon \rightarrow 0 \quad (4.4)$$

uniformly, with $a_0, a_1, a_2, \dots \in C^\infty(M \times M)$ some Riemannian invariants such that $a_0(x, x) = 1$ as proved for instance in [6]. We have also a uniform bound : there exists $A_0 > 1$ such that for any $\epsilon > 0$,

$$\forall x, y \in M, \frac{1}{A_0 4\pi\epsilon} e^{-\frac{d_g(x, y)^2}{4\epsilon}} \leq p_\epsilon(x, y) \leq \frac{A_0}{4\pi\epsilon} e^{-\frac{d_g(x, y)^2}{4\epsilon}}. \quad (4.5)$$

We deduce the same uniform properties for the rescaled heat kernel $\hat{p}_\epsilon(x, y)$ by some parameters $a_\epsilon \in \mathbb{R}^2$ and $\alpha_\epsilon > 0$ such that $a_\epsilon \rightarrow a \in \mathbb{R}^2$ and $\alpha_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. We have for any $R > 0$,

$$\hat{p}_\epsilon(z, y) = \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+o(1))}}{4\pi\theta_\epsilon} (1 + o(1)) \text{ uniformly on } \mathbb{D}_R \times \mathbb{D}_R, \quad (4.6)$$

where $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)} \alpha_\epsilon^2}$ and we have the following bound for any fixed $0 < \rho < 1$

$$\frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+\rho)}}{4\pi\theta_\epsilon} (1 - \rho) \leq \hat{p}_\epsilon(z, y) \leq \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1-\rho)}}{4\pi\theta_\epsilon} (1 + \rho) \quad (4.7)$$

for all $\epsilon > 0$ small enough.

Let's prove (4.6). We fix $R > 0$ and we have uniformly for $(x, y) \in \mathbb{D}_R \times \mathbb{D}_R$ as $\epsilon \rightarrow 0$

$$\begin{aligned} \hat{p}_\epsilon(x, y) &= \frac{\alpha_\epsilon^2 e^{2\tilde{v}_l(\check{x})}}{4\pi\epsilon} e^{-\frac{d_g(\check{x}, \check{y})^2}{4\epsilon}} (a_0(\check{x}, \check{y}) + o(1)) \\ &= \frac{\alpha_\epsilon^2 e^{2\tilde{v}_l(a)}}{4\pi\epsilon} (1 + o(1)) e^{-\frac{d_g(\check{x}, \check{y})^2}{4\epsilon}} \end{aligned}$$

by (4.4). We have that

$$d_g(\check{x}, \check{y}) = e^{\tilde{v}_l(a)} |x - y| \alpha_\epsilon (1 + o(1))$$

uniformly for $(x, y) \in \mathbb{D}_R \times \mathbb{D}_R$. This leads to the desired approximation (4.6).

For a sequence of measures $\nu_\epsilon \in \mathcal{M}(M)$, we also have uniform bounds for $R > r > 0$ and $\theta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$:

$$\sup_{x \in \mathbb{D}_{R-r}} \int_{M \setminus \mathbb{D}_R} \alpha_\epsilon^2 p_\epsilon(\check{x}, y) d\nu_\epsilon(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\theta_\epsilon}\right). \quad (4.8)$$

We prove it thanks to (4.5) and (4.7). Let $x \in \mathbb{D}_{R-r}$ and let us write that

$$\begin{aligned}
\alpha_\epsilon^2 \int_{M \setminus \check{\mathbb{D}}_R} p_\epsilon(\check{x}, y) dv_\epsilon(y) &= e^{-2v_l(\check{x})} \int_{\mathbb{D}_{C_0^2 R} \setminus \mathbb{D}_R} \hat{p}_\epsilon(x, z) d\hat{v}_\epsilon(z) \\
&\quad + \int_{M \setminus \check{\mathbb{D}}_{C_0^2 R}} \alpha_\epsilon^2 p_\epsilon(\check{x}, y) dv_\epsilon(y) \\
&\leq C_0^2 \int_{\mathbb{D}_{C_0^2 R} \setminus \mathbb{D}_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{2\pi\theta_\epsilon} d\hat{v}_\epsilon(z) \\
&\quad + \int_{M \setminus B_g(\bar{a}_\epsilon, \frac{\alpha_\epsilon C_0^2 R}{C_0})} \frac{\alpha_\epsilon^2 A_0}{4\pi\epsilon} e^{-\frac{d_g(\check{x}, y)^2}{4\epsilon}} dv_\epsilon(y) \\
&\leq O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\theta_\epsilon}\right) + \frac{A_0 \alpha_\epsilon^2}{4\pi\epsilon} e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}},
\end{aligned}$$

where $\check{\mathbb{D}}_r \subset B_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r) \subset B_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 R)$. This proves (4.8). We also have that

$$\sup_{x \in M \setminus \check{\mathbb{D}}_R} \int_{\check{\mathbb{D}}_r} p_\epsilon(x, y) dv_g(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\theta_\epsilon}\right). \quad (4.9)$$

Let $x \in M \setminus \check{\mathbb{D}}_R$. We assume that $x \in \mathbb{D}_{C_0^2 R} \setminus \mathbb{D}_R$. We write that

$$\int_{\check{\mathbb{D}}_r} p_\epsilon(x, y) dv_g(y) = \int_{\mathbb{D}_r} \hat{p}_\epsilon(z, \check{x}) dz \leq \frac{1}{2\pi\theta_\epsilon} \int_{\mathbb{D}_r} e^{-\frac{|x-z|^2}{8\theta_\epsilon}} dz \leq \frac{r^2}{\theta_\epsilon} e^{-\frac{(R-r)^2}{8\theta_\epsilon}}$$

if ϵ is small enough and if $x \in M \setminus \check{\mathbb{D}}_{C_0^2 R} \subset M \setminus B_g(\bar{a}_\epsilon, \alpha_\epsilon R C_0)$, we write that

$$\begin{aligned}
\int_{\check{\mathbb{D}}_r} p_\epsilon(x, y) dv_g(y) &\leq \int_{B_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} p_\epsilon(x, y) dv_g(y) \\
&\leq \frac{A_0}{4\pi\epsilon} \int_{B_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} e^{-\frac{d_g(x, y)^2}{4\epsilon}} dv_g(y) \\
&\leq O\left(\frac{e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}}}{\theta_\epsilon}\right).
\end{aligned}$$

This proves 4.9. Now let's prove that

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \left| \int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0. \quad (4.10)$$

We fix $0 < \rho < \frac{1}{2}$ and $R > 0$. Then for ϵ small enough, we have by (4.7) that

$$\int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz \leq \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-z|^2(1-\rho)}{4\theta_\epsilon}}}{4\pi\theta_\epsilon} (1+\rho) dz = \frac{1+\rho}{1-\rho}$$

for any $x \in \mathbb{D}_r$ and that

$$\begin{aligned} \int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz &\geq \int_{\mathbb{D}_R} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{4\pi\theta_\epsilon} (1-\rho) dz \\ &\geq \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{4\pi\theta_\epsilon} (1-\rho) dz - \int_{\mathbb{R}^2 \setminus \mathbb{D}_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{2\pi\epsilon} dz \\ &\geq \frac{1-\rho}{1+\rho} + o(1) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

uniformly on \mathbb{D}_r . Letting $\epsilon \rightarrow 0$, then $R \rightarrow +\infty$ and then $\rho \rightarrow 0$ gives (4.10).

4.2.3 Poincaré inequalities and capacity

We first notice the following consequence of the classical computation of the capacity of annuli in \mathbb{R}^2 .

Claim 18. *Let (M, g) be a compact Riemannian surface. Then, there is $C > 0$ and $r_0 > 0$ such that for all $x \in M$ and all $0 < r_2 < r_1 < r_0$, there exists a smooth function $\eta_{g,x,r_1,r_2} : M \rightarrow \mathbb{R}$ with*

- $0 \leq \eta_{g,x,r_1,r_2} \leq 1$
- $\eta_{g,x,r_1,r_2} = 1$ on $B_g(x, r_2)$
- $\eta_{g,x,r_1,r_2} \in C_c^\infty(B_g(x, r_1))$
- $\int_M |\nabla \eta_{g,x,r_1,r_2}|_g^2 dv_g \leq \frac{C}{\ln(\frac{r_1}{r_2})}$.

We now recall two theorems giving Poincaré inequalities on surfaces.

Theorem 10 ([1], Lemma 8.3.1). *Let (M, g) be a Riemannian manifold. Then, there exists a constant $B > 0$ such that for any $L \in W^{-1,2}(M)$ with $L(1) = 1$, we have the following Poincaré inequality*

$$\forall f \in W^{1,2}(M), \int_M (f - L(f))^2 dv_g \leq B \|L\|_{W^{-1,2}(M)}^2 \int_M |\nabla f|_g^2 dv_g.$$

We denote by

$$C_{1,2}(K) = \inf \left\{ \int_{\mathbb{R}^2} \phi^2 dv_g + \int_{\mathbb{R}^2} |\nabla \phi|_g^2 dv_g; \phi \in C_c^\infty(\mathbb{R}^2), \phi \geq 1 \text{ on } K \right\}$$

the capacity of a compact set $K \subset \mathbb{R}^2$ and

$$Cap_2(K, \Omega) = \inf \left\{ \int_\Omega |\nabla \phi|_g^2 dv_g; \phi \in C_c^\infty(\Omega), \phi \geq 1 \text{ on } K \right\}$$

the relative capacity of $K \subset \subset \Omega$.

Theorem 11 ([1], Corollary 8.2.2). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then, there exists a constant C_Ω such that for any compact $K \subset \Omega$ with $C_{1,2}(K) > 0$ and for any function $f \in C^\infty(\Omega)$ such that $f = 0$ on K ,*

$$\|f\|_{L^2(\Omega)} \leq \frac{C_\Omega}{C_{1,2}(K)} \|\nabla f\|_{L^2(\Omega)}.$$

Ω is a bounded extension domain means that the extension by 0 on \mathbb{R}^2 of every function in $W_0^{1,2}(\Omega)$ is $W^{1,2}$ in \mathbb{R}^2 . This is true for the family of sets we consider during the proof :

$$\Omega = \mathbb{D}_{\frac{1}{\rho}} \setminus \bigcup_{i=1}^s \mathbb{D}_{\rho}(x_i) ,$$

where $\rho > 0$, $x_i \in \mathbb{D}_{\frac{1}{\rho}}$ such that if $i \neq j$, then $x_i \neq x_j$ and

$$10\rho < \min \left(\min_i d(x_i, \partial \mathbb{D}_{\frac{1}{10\rho}}); \min_{i \neq j} \frac{|x_i - x_j|}{2} \right) .$$

We now set

$$\Omega_K = \mathbb{D}_{\frac{1}{K\rho}} \setminus \bigcup_{i=1}^s \mathbb{D}_{K\rho}$$

for some fixed number $1 < K < 10$ chosen independent of the problem we consider. We obtain the corollary :

Corollary. *Let $r > 0$ fixed. Then, we have a constant $C_r > 0$ such that for every $f \in \mathcal{C}^\infty(\Omega)$ which vanishes on a smooth piecewise curve $\Gamma \subset \subset \Omega_K$ which connects two points of distance $r > 0$,*

$$\|f\|_{L^2(\Omega)} \leq C_r \|\nabla f\|_{L^2(\Omega)} .$$

Indeed, it is proved in ([52], pages 95-97) that

$$Cap_2(\Gamma, \Omega) \geq \frac{K_0}{\ln(\frac{1}{r})}$$

and that

$$C_{1,2}(\Gamma) \geq K_1 Cap_2(\Gamma, \Omega)$$

for constants $K_0 > 0$ and $K_1 > 0$ which only depend on Ω and K .

4.3 Selection of a maximizing sequence

Let (M, g) be a smooth compact Riemannian surface without boundary of unit volume. We fix $k \geq 1$. As in [91] for the maximization of the first non-zero eigenvalue, we build a specific maximizing sequence for $\Lambda_k(M, [g])$ thanks to the heat operator. Let $\epsilon > 0$. We denote by K_ϵ the heat operator so that for a positive Radon measure $\nu \in \mathcal{M}(M)$, $K_\epsilon[\nu]dv_g$ is the solution at time $\epsilon > 0$ of the heat equation on the Riemannian surface (M, g) which converges to ν as $\epsilon \rightarrow 0$ in $\mathcal{M}(M)$. Given $x, y \in M$, we denote by $p_\epsilon(x, y)$ the heat kernel of (M, g) so that for $\nu \in \mathcal{M}(M)$,

$$K_\epsilon[\nu](x) = \int_M p_\epsilon(x, y) d\nu(y) .$$

For $f \in L^1(M)$, we let $K_\epsilon[f] := K_\epsilon[f dv_g]$ so that

$$\int_M K_\epsilon[f] d\nu = \int_M f K_\epsilon[\nu] dv_g .$$

For $\epsilon > 0$, we set

$$\lambda_\epsilon = \sup_{\nu \in \mathcal{M}_1(M)} \lambda_k(K_\epsilon[\nu]g) \quad (4.11)$$

so that by continuity of $\nu \in \mathcal{M}_1(M) \mapsto \lambda_k(K_\epsilon[\nu]g)$, a maximizing sequence for the variational problem (4.11) converges in $\mathcal{M}_1(M)$, up to the extraction of a subsequence, to a measure $\nu_\epsilon \in \mathcal{M}_1(M)$ such that

$$\lambda_\epsilon = \lambda_k(K_\epsilon[\nu_\epsilon]g).$$

We set $e^{2u_\epsilon} = K_\epsilon[\nu_\epsilon]$. This sequence of smooth positive functions $\{e^{2u_\epsilon}\}_{\epsilon>0}$ defines a maximizing sequence for $\Lambda_k(M, [g])$ as $\epsilon \rightarrow 0$.

Indeed, $\lambda_\epsilon \leq \Lambda_k(M, [g])$ for all $\epsilon > 0$ and if $\eta > 0$, we have some metric $\tilde{g} = e^{2u}g \in [g]$ such that $\text{Vol}_{\tilde{g}} = 1$ and $\lambda_k(\tilde{g}) \geq \Lambda_k(M, [g]) - \frac{\eta}{2}$. By definition of the heat operator, $K_\epsilon[dv_{\tilde{g}}] = K_\epsilon[e^{2u}]$ and by the estimate (4.4) in Section 4.2.2, we have $K_\epsilon[e^{2u}] \rightarrow e^{2u}$ as $\epsilon \rightarrow 0$ in $\mathcal{C}^0(M)$. Then there exists $\epsilon_0 > 0$ such that

$$\lambda_\epsilon \geq \lambda_k(K_\epsilon[dv_{\tilde{g}}]) \geq \lambda_k(\tilde{g}) - \frac{\eta}{2} \geq \Lambda_k(M, [g]) - \eta$$

for $\epsilon < \epsilon_0$. This proves that $\lambda_\epsilon \rightarrow \Lambda_k(M, [g])$ as $\epsilon \rightarrow 0$.

We first obtain an Euler-Lagrange characterization associated to the maximization problem (4.11).

Proposition 2. *Fix $\epsilon > 0$. Then, there exists a family $\Phi_\epsilon = (\phi_\epsilon^1, \dots, \phi_\epsilon^{n(\epsilon)+1})$ of smooth functions, independent in $L^2(M, e^{2u_\epsilon}g)$, such that*

- (i) $\forall i \in \{1, \dots, n(\epsilon) + 1\}, \Delta_g \phi_\epsilon^i = \lambda_\epsilon e^{2u_\epsilon} \phi_\epsilon^i,$
- (ii) $K_\epsilon[|\Phi_\epsilon|^2] \geq 1$ on M ,
- (iii) $K_\epsilon[|\Phi_\epsilon|^2] = 1$ on $\text{supp}(\nu_\epsilon)$.

Note that the proof is the same as in [91], Claim 5, and that the number $n(\epsilon) + 1$ of independent eigenfunctions can be chosen independent of ϵ . Indeed, there exists a bound on the multiplicity of k -th eigenvalues on surfaces which only depends on k and the genus of the surface (see [15]). Up to the extraction of a subsequence, we thus assume in the following that $n(\epsilon) = n$ is fixed.

We now denote by ν the weak* limit of $\{e^{2u_\epsilon} dv_g\}$ as $\epsilon \rightarrow 0$. Notice that ν is also the weak* limit of $\{\nu_\epsilon\}$. Indeed, if $\zeta \in \mathcal{C}^0(M)$,

$$\begin{aligned} \left| \int_M \zeta (e^{2u_\epsilon} dv_g - d\nu_\epsilon) \right| &= \left| \int_M (K_\epsilon[\zeta] - \zeta) d\nu_\epsilon \right| \\ &\leq \sup_M |K_\epsilon[\zeta] - \zeta| \end{aligned}$$

which goes to 0 as $\epsilon \rightarrow 0$ by uniform continuity of ζ .

We aim at proving that ν is absolutely continuous with respect to dv_g with a smooth density. We organize the proof of Theorem 9 as follows :

In Section 4.4, we give regularity estimates on the densities e^{2u_ϵ} and the associated eigenfunctions ϕ_ϵ^i defined by Proposition 2 (see Claim 21). These estimates permit to pass to the

limit on the eigenvalue equation (Proposition 2 (i)) as $\epsilon \rightarrow 0$ (see Claim 22). However, we cannot pass to the limit on the whole surface. We have to avoid some singularities for the maximizing sequence which could occur. In the general case $k \geq 1$, we cannot remove a priori some concentration points of $\{e^{2u_\epsilon} dv_g\}$ even with the assumption that (4.3) is strict. Note that in the case $k = 1$, this same assumption permits directly to rule out the appearance of these concentration points (see Petrides [91]) Other harmless singularities are also carefully avoided (see Claim 20).

From Sections 4.5 to 4.7, we assume the existence of concentration points for the maximizing sequence and we aim at deducing the case of equality in (4.3). In Section 4.5, we detect all the concentration scales thanks to the construction of a bubble tree. This leads to the proof of Proposition 3, page 107.

We then give in Section 4.6 regularity estimates on the eigenfunctions at each scale of concentration and pass to the limit in the equation they satisfy. Notice that this work is divided into two subsections, depending on the speed of convergence to zero of the concentration scale α_ϵ as $\epsilon \rightarrow 0$.

Finally, in Section 4.7.1, capitalizing on the energy estimates for the limiting measures and equations given in Section 4.4.2 on M (see (4.30)), at the end of Section 4.6.1 (see (4.76)) and Section 4.6.2 (see (4.81)) on some spheres S^2 , we both prove the regularity of the limiting measures at all the scales of concentration, and that no energy is lost in the necks in the bubbling process. This is given by Proposition 4, page 142. Thanks to this proposition, we prove in Section 4.7.2 that the presence of concentration points imply the case of equality in (4.3) by a suitable choice of test functions for the variational characterization of $\lambda_\epsilon = \lambda_k(e^{2u_\epsilon} g)$.

Therefore, since the specific maximizing sequence $\{e^{2u_\epsilon} dv_g\}$ does not concentrate with the assumption that (4.3) is strict, the proof of Theorem 9 just uses the second part of Proposition 4 in Section 4.7.1. Notice that in the case $k = 1$ on surfaces which are not diffeomorphic to the sphere, we already know that any maximizing sequence does not concentrate as proved in [91] since we have $\Lambda_1(\Sigma, [g]) > 8\pi$. Thus, in this case, we did not need the multi-bubble asymptotic analysis of Sections 4.5 and 4.6.

4.4 Regularity estimates on the surface

We refer to Section 4.2.1 for the notations, in particular in the charts of computation on the fixed metric g .

4.4.1 Regularity estimates far from singularities

In this subsection, we will adapt the arguments used in [91], Section 2.3, in order to get finer and finer estimates on the eigenfunctions which appear in Proposition 2, and pass to the limit on the equation they satisfy.

We first get, by point (iii) in Proposition 2, uniform estimates on the eigenfunctions $\{\phi_i^\epsilon\}$ on sets of points which lie at a distance to $\text{supp}(v_\epsilon)$ asymptotically smaller than $\sqrt{\epsilon}$.

Claim 19. *For any $R > 0$, there exists a constant $C_R > 0$ such that for any sequence (x_ϵ) of points in M with $d_g(x_\epsilon, \text{supp}(v_\epsilon)) \leq R\sqrt{\epsilon}$, we have*

$$|\phi_\epsilon^i(x_\epsilon)| \leq C_R \text{ for all } \epsilon > 0.$$

Proof. We refer the reader to Section 4.2.1 for the notations used during this proof. We can assume that $x_\epsilon \in \omega_l$ for $1 \leq l \leq L$ fixed and we set

$$\hat{\Phi}_\epsilon(x) = \tilde{\Phi}_\epsilon^l \left(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l \right)$$

for $x \in \mathbb{D}_{\frac{\delta}{\sqrt{\epsilon}}}$. Then

$$\Delta_{\tilde{g}} \hat{\Phi}_\epsilon^i = \epsilon \lambda_\epsilon e^{2\tilde{u}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)} \hat{\Phi}_\epsilon^i$$

in $\mathbb{D}_{\delta\sqrt{\epsilon}}$ for $1 \leq i \leq n+1$. By estimate (4.5) of Section 4.2.2, (ϵp_ϵ) is uniformly bounded so that $\left(\epsilon e^{2\tilde{u}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)} \right)$ is uniformly bounded.

Now, let $y_\epsilon \in \text{supp}(v_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) \leq R\sqrt{\epsilon}$. Thanks to Proposition 2, we have that $K_\epsilon \left[|\Phi_\epsilon|^2 \right] (y_\epsilon) = 1$. Let us write then with (4.5), in Section 4.2.2, that for $\rho > 0$,

$$\begin{aligned} 1 = K_\epsilon \left[|\Phi_\epsilon|^2 \right] (y_\epsilon) &\geq \sum_{i=1}^{n+1} K_\epsilon \left[|\phi_\epsilon^i|^2 \right] (y_\epsilon) \\ &= \sum_{i=1}^{n+1} \int_M p_\epsilon(y, y_\epsilon) \left(\phi_\epsilon^i(y) \right)^2 dv_g(y) \\ &\geq \sum_{i=1}^{n+1} \frac{1}{4\pi A_0 \epsilon} e^{-\rho^2 C_0^2} \int_{B_g(y_\epsilon, 2\rho C_0 \sqrt{\epsilon})} \left(\phi_\epsilon^i(y) \right)^2 dv_g(y) \\ &\geq \sum_{i=1}^{n+1} \frac{1}{4\pi A_0 C_0^2} e^{-\rho^2 C_0^2} \int_{\mathbb{D}_{2\rho}(\hat{z}_\epsilon)} \left(\hat{\Phi}_\epsilon^i(z) \right)^2 dz. \end{aligned}$$

We set $\hat{z}_\epsilon = \frac{1}{\sqrt{\epsilon}}(\tilde{y}_\epsilon - \tilde{x}_\epsilon)$ so that, up to a subsequence, $\hat{z}_\epsilon \rightarrow z_0$ as $\epsilon \rightarrow 0$ and we deduce from the previous inequality that, for any $\rho > 0$, $\{\hat{\Phi}_\epsilon^i\}$ is bounded in $L^2(\mathbb{D}_\rho(z_0))$. Thus, by the Sobolev embedding $W^{2,2} \subset C^0$ (see [43], Corollary 7.11, page 158) and the L^2 elliptic estimate (see [43], Theorem 9.11, page 235), it is clear that $\{\hat{\Phi}_\epsilon\}$ is uniformly bounded in \mathbb{D}_ρ by some constant D_ρ . Setting $C_R = D_{2C_0R}$ gives the claim. \diamond

Now, we aim at locating all the singularities for the maximizing sequence $\{e^{2u_\epsilon} dv_g\}$ which could appear, on small balls around a finite number of points, in order to continue the estimates far from these points. Let's formulate the singular properties that a point $x \in (M, g)$ could satisfy, for $r > 0$ and $\epsilon > 0$:

$$\mathbf{A}_{r,\epsilon} \quad \lambda_\star(B_g(x, r), e^{2u_\epsilon} g) \leq \frac{\Lambda_k(M, [g])}{2}.$$

$\mathbf{B}_{r,\epsilon}$ There exists $f \in E_k(e^{2u_\epsilon} g)$ such that $f(x) = 0$ and the Nodal set of f which contains x does not intersect $\partial B_g(x, r)$.

We say that x satisfies $\mathbf{P}_{r,\epsilon}$ if it satisfies $\mathbf{A}_{r,\epsilon}$ or $\mathbf{B}_{r,\epsilon}$. Note that if $r_1 < r_2$, then $\mathbf{A}_{r_1,\epsilon} \Rightarrow \mathbf{A}_{r_2,\epsilon}$ and $\mathbf{B}_{r_1,\epsilon} \Rightarrow \mathbf{B}_{r_2,\epsilon}$. For a manifold M , a sequence of densities $\{e^{2u_\epsilon}\}$ and $r > 0$, we define the singular set :

$$X_r(M, \{e^{2u_\epsilon} g\}) = \{x \in M; \text{there exists } \epsilon > 0 \text{ such that } x \text{ satisfies } \mathbf{P}_{r,\epsilon}\}. \quad (4.12)$$

Note that if $r_1 < r_2$, then $X_{r_1}(M, \{e^{2u_\epsilon} g\}) \subset X_{r_2}(M, \{e^{2u_\epsilon} g\})$. The following claim holds true :

Claim 20. *There exists a subsequence $\{e^{2u_{\epsilon_j}}g\}_{j \geq 0}$ with $\epsilon_j \rightarrow 0$ as $j \rightarrow +\infty$ and there exist s points $p_1, \dots, p_s \in M$ with $0 \leq s \leq k$ such that*

- $\forall \rho > 0, \exists r > 0, X_r(M, \{e^{2u_{\epsilon_j}}g\}_{j \geq 0}) \subset \bigcup_{i=1}^s B_g(p_i, \rho),$
- *For any subsequence $\{e^{2u_{\epsilon_{j(m)}}}g\}_{m \geq 0}$ of $\{e^{2u_{\epsilon_j}}g\}_{j \geq 0},$*

$$\forall \rho > 0, \forall r > 0, \forall 1 \leq i \leq s, X_r \left(M, \{e^{2u_{\epsilon_{j(m)}}}g\}_{m \geq 0} \right) \cap B_g(p_i, \rho) \neq \emptyset. \quad (4.13)$$

Proof. Assume by contradiction that for any sequence $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$, for any series of s points p_1, \dots, p_s with $s \leq k$, there exists $\rho > 0$ such that for all $r > 0$,

$$X_r(M, \{e^{2u_{\epsilon_j}}g\}_{j \geq 0}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho) \neq \emptyset. \quad (4.14)$$

Thanks to this hypothesis, we will deduce by induction the following property \mathbf{H}_s for $1 \leq s \leq k+1$:

\mathbf{H}_s : there exist sequences $\epsilon_j \rightarrow 0$ and $r_j \searrow 0$ as $j \rightarrow \infty$ and s distinct points p_1^j, \dots, p_s^j and p_1, \dots, p_s such that for all $1 \leq i \leq s$, $p_i^j \rightarrow p_i$ as $j \rightarrow \infty$ and such that for all $j \geq 0$, p_i^j satisfies $\mathbf{P}_{r_j, \epsilon_j}$.

Let's first prove \mathbf{H}_1 . We apply (4.14) for $s = 0$ and for a sequence 2^{-m} , so that for a fixed $j \in \mathbb{N}$, there exists $p_1^j \in X_{2^{-j}}(M, \{e^{2u_{2^{-m}}}dv_g\}_{m \geq 0})$. For $j \geq 0$, we choose $\epsilon_j = 2^{-m(j)}$ such that p_1^j satisfies $\mathbf{P}_{2^{-j}, \epsilon_j}$. It is clear that $\epsilon_j \rightarrow 0$. Up to the extraction of a subsequence, there is $p_1 \in M$ such that $p_1^j \rightarrow p_1$ as $j \rightarrow \infty$ and we get \mathbf{H}_1 .

We assume now that \mathbf{H}_s is true for some $1 \leq s \leq k$. We consider the sequences $\{\epsilon_j\}$, $\{r_j\}$, $\{p_i^j\}$ and the points $\{p_1, \dots, p_s\}$ given by \mathbf{H}_s . Let us prove \mathbf{H}_{s+1} . By (4.14), there is $\rho > 0$ such that for all $r > 0$,

$$X_r(M, \{e^{2u_{\epsilon_j}}g\}_{j \geq 0}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho) \neq \emptyset.$$

Let $p_{s+1}^j \in X_{r_j}(M, \{e^{2u_{\epsilon_j}}g\}_{j \geq 0}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$. For $j \geq 0$, we let $\alpha(j)$ be such that p_{s+1}^j satisfies $\mathbf{P}_{r_j, \epsilon_{\alpha(j)}}$. Since $r_j \rightarrow 0$ as $j \rightarrow +\infty$, it is clear that $\alpha(j) \rightarrow +\infty$. We set $m(j) = \min\{j, \alpha(j)\}$. By \mathbf{H}_s , for $1 \leq i \leq s$, $p_i^{\alpha(j)}$ satisfies $\mathbf{P}_{r_{\alpha(j)}, \epsilon_{\alpha(j)}}$ and since r_j is decreasing, $p_i^{\alpha(j)}$ satisfies $\mathbf{P}_{r_{m(j)}, \epsilon_{\alpha(j)}}$. Moreover, p_{s+1}^j satisfies $\mathbf{P}_{r_j, \epsilon_{\alpha(j)}}$ and since r_j is decreasing, p_{s+1}^j satisfies $\mathbf{P}_{r_{m(j)}, \epsilon_{\alpha(j)}}$. Up to the extraction of a subsequence, we can suppose that $r_{m(j)} \searrow 0$ as $j \rightarrow \infty$ and we let $p_{s+1} \in M$ such that $p_{s+1}^j \rightarrow p_{s+1}$ as $j \rightarrow +\infty$. Since $p_{s+1}^j \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$, $p_{s+1} \notin \{p_1, \dots, p_s\}$. This proves \mathbf{H}_{s+1} .

The proof of \mathbf{H}_{k+1} is complete. Now, we prove that \mathbf{H}_{k+1} leads to a contradiction. We define $k+1$ test functions for the variational characterization (4.2.1) of $\lambda_{\epsilon_j} = \lambda_k(e^{2u_{\epsilon_j}}g)$, η_i^j for $j \in \mathbb{N}$ and $1 \leq i \leq k+1$, as follows :

- If p_i^j satisfies $\mathbf{A}_{r_j, \epsilon_j}$, η_i^j is an eigenfunction for $\lambda_\star(B_g(p_i^j, r_j), e^{2u_{\epsilon_j}}g)$ extended by zero in $M \setminus B_g(p_i^j, r_j)$. In this case,

$$\frac{\int_M |\nabla \eta_i^j|_g^2 dv_g}{\int_M (\eta_i^j)^2 dv_g} \leq \frac{\Lambda_k(M, [g])}{2}. \quad (4.15)$$

- If p_i^j does not satisfy $\mathbf{A}_{r_j, \epsilon_j}$, it satisfies $\mathbf{B}_{r_j, \epsilon_j}$ and in this case, η_i^j is some eigenfunction for $\lambda_\star(D_i^j, e^{2u_{\epsilon_j}}g)$ extended by zero in $M \setminus D_i^j$ where D_i^j is a Nodal domain of some eigenfunction associated to λ_{ϵ_j} , which is included in $B_g(p_i^j, r_j)$. Such a domain exists by the assumption $\mathbf{B}_{r_j, \epsilon_j}$. In this case,

$$\frac{\int_M |\nabla \eta_i^j|_g^2 dv_g}{\int_M (\eta_i^j)^2 dv_g} = \lambda_\star(D_i^j, e^{2u_{\epsilon_j}}g) = \lambda_{\epsilon_j}. \quad (4.16)$$

For j large enough, we have

$$\min_{1 \leq i < i' \leq k+1} d_g(p_i^j, p_{i'}^j) - 3r_j \geq \frac{1}{2} \min_{1 \leq i < i' \leq k+1} d_g(p_i, p_{i'}) > 0$$

so that the functions $\eta_1^j, \dots, \eta_{k+1}^j$ have pairwise disjoint supports. Thanks to (4.15) and (4.16), the min-max characterization of $\lambda_{\epsilon_j} = \lambda_k(e^{2u_{\epsilon_j}}g)$ (4.2.1) gives that

$$\lambda_{\epsilon_j} \leq \max_{1 \leq i \leq k+1} \frac{\int_M |\nabla \eta_i^j|_g^2 dv_g}{\int_M (\eta_i^j)^2 dv_g} \leq \lambda_{\epsilon_j}$$

since for j large enough, $\lambda_{\epsilon_j} \rightarrow \Lambda_k(M, [g]) > \frac{\Lambda_k(M, [g])}{2}$. Then, all the inequalities are equalities and by the case of equality in the min-max characterization of the k -th eigenvalue, one of the functions η_i^j is an eigenfunction on the manifold for $\lambda_{\epsilon_j} = \lambda_k(M, e^{2u_{\epsilon_j}}g)$. Since $\text{supp}(\eta_i^j) \subset B_g(p_i^j, r_j)$, this contradicts the maximum principle.

Therefore, we have proved that there exist a subsequence of $\{e^{2u_{\epsilon_j}}dv_g\}_{j \geq 0}$ and p_1, \dots, p_s for some $0 \leq s \leq k$ such that

$$\forall \rho > 0, \exists r > 0, X_r(M, \{e^{2u_{\epsilon_j}}g\}_{j \geq 1}) \subset \bigcup_{i=1}^s B_g(p_i, \rho),$$

which is exactly the first part of the claim.

Let's prove now the second part of the claim. If there exists a subsequence $j(m) \rightarrow +\infty$ as $m \rightarrow +\infty$ such that there exists $\rho > 0, r > 0$ and $1 \leq i_0 \leq s$ with

$$X_r(M, \{e^{2u_{\epsilon_{j(m)}}}g\}_{m \geq 0}) \cap B_g(p_{i_0}, \rho) = \emptyset,$$

then, taking the subsequence $j(m)$, we can remove the index $i_0 \in \{1, \dots, s\}$ so that

$$X_r(M, \{e^{2u_{\epsilon_j(m)}} g\}) \subset \bigcup_{i \in \{1, \dots, s\} \setminus \{i_0\}} B_g(p_i, \rho).$$

We go on with this process until we cannot find a subsequence such that (4.13) does not hold. This ends the proof of the claim. \diamond

Up to the extraction of a subsequence, we assume in the following that $\{e^{2u_\epsilon} g\}$ satisfies the conclusion of Claim 20. For $\rho > 0$, we let

$$M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho).$$

We are now able to get regularity estimates on the functions e^{2u_ϵ} and Φ_ϵ in $M(\rho)$.

Claim 21. *We assume that $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} e^{2u_\epsilon} dv_g > 0$ for any $\rho > 0$ small enough. Then we have the following :*

— *Estimates on Φ_ϵ :*

$$\forall \rho > 0, \exists C_1(\rho) > 0, \forall \epsilon > 0, \|\Phi_\epsilon\|_{W^{1,2}(M(\rho))} \leq C_1(\rho), \quad (4.17)$$

$$\forall \rho > 0, \exists C_2(\rho) > 0, \forall \epsilon > 0, \|\Phi_\epsilon\|_{C^0(M(\rho))} \leq C_2(\rho). \quad (4.18)$$

— *Quantitative non-concentration estimates on e^{2u_ϵ} and $|\nabla \Phi_\epsilon|_g^2$:*

$$\forall \rho > 0, \exists D_1(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in M(\rho)} \int_{B_g(x, r)} e^{2u_\epsilon} dv_g \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})}, \quad (4.19)$$

$$\forall \rho > 0, \exists D_2(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in M(\rho)} \int_{B_g(x, r)} |\nabla \Phi_\epsilon|_g^2 dv_g \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}}. \quad (4.20)$$

Proof. We first prove (4.17) by using Claim 20 and the assumption $m_0(\rho) > 0$.

For that purpose, let's prove that $\{\frac{e^{2u_\epsilon}}{\int_{M(\rho)} e^{2u_\epsilon} dv_g}\}$ is bounded in $W^{-1,2}(M(\rho))$. Let $\rho > 0$ and let $r > 0$ be such that $X_r(M, \{e^{2u_\epsilon} g\}) \subset \bigcup_{i=1}^s B_g(p_i, \rho)$. Then, for all $x \in M(\rho)$ and all $\epsilon > 0$, $\lambda_\star(B_g(x, r), e^{2u_\epsilon} g) > \frac{\Lambda_k(M, [g])}{2}$. By the compactness of $M(\rho)$, we can find $y_1, \dots, y_t \in M(\rho)$ such that

$$M(\rho) \subset \bigcup_{i=1}^t B_g(y_i, r).$$

Let ψ_1, \dots, ψ_t be a partition of unity associated to this covering, so that $\sum_{i=1}^t \psi_i = 1$ on $M(\rho)$ and $\text{supp}(\psi_i) \subset B_g(y_i, r)$. Let $L : W^{1,2}(M(\rho)) \rightarrow W^{1,2}(M)$ be a continuous extension operator.

Then, if $\psi \in W^{1,2}(M(\rho))$,

$$\begin{aligned}
 \int_{M(\rho)} \psi \frac{e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} &= \sum_{i=1}^t \int_{M(\rho) \cap B_g(y_i, r)} \psi \psi_i \frac{e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} \\
 &\leq \sum_{i=1}^t \left(\int_{M(\rho) \cap B_g(y_i, r)} (\psi_i \psi)^2 \frac{e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^t \left(\int_{B_g(y_i, r)} (\psi_i L(\psi))^2 \frac{e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^t \frac{\left(\int_{B_g(y_i, r)} |\nabla(\psi_i L(\psi))|_g^2 dv_g \right)^{\frac{1}{2}}}{\lambda_\star(B_g(y_i, r), e^{2u_\epsilon} g)^{\frac{1}{2}} \left(\int_{M(\rho)} e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}}} \\
 &\leq \frac{A_0(\rho)}{\left(\frac{\Lambda_k(M, [g])}{2} \right)^{\frac{1}{2}} m_0(\rho)^{\frac{1}{2}}} \|L(\psi)\|_{W^{1,2}(M)} \\
 &\leq A_1(\rho) \|\psi\|_{W^{1,2}(M(\rho))}
 \end{aligned}$$

for some constants $A_0(\rho)$ and $A_1(\rho)$ which do not depend on $\epsilon > 0$.

By Theorem 10 in Section 4.2.3, we now get the following Poincaré inequality : there exists some constant $A_2(\rho)$ such that for any $f \in C^\infty(M(\rho))$

$$\forall \epsilon > 0, \int_{M(\rho)} \left(f - \int_{M(\rho)} f \frac{e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} \right)^2 dv_g \leq A_2(\rho) \int_{M(\rho)} |\nabla f|_g^2 dv_g.$$

We deduce from this inequality that

$$\|f\|_{L^2(M(\rho))} \leq \left(A_2(\rho) \int_M |\nabla f|_g^2 dv_g + \frac{\int_M f^2 e^{2u_\epsilon} dv_g}{\int_{M(\rho)} e^{2u_\epsilon} dv_g} \right)^{\frac{1}{2}} + \frac{\left(\int_M f^2 e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}}}{\left(\int_{M(\rho)} e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}}}$$

Since $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} e^{2u_\epsilon} dv_g > 0$, applying this inequality to ϕ_ϵ^i , together with the fact that

$$\int_M |\nabla \phi_\epsilon^i|_g^2 dv_g = \lambda_\epsilon \int_M e^{2u_\epsilon} (\phi_\epsilon^i)^2 dv_g$$

and that, by (iii) of Proposition 2,

$$\int_M (\phi_\epsilon^i)^2 e^{2u_\epsilon} dv_g \leq \int_M |\Phi_\epsilon|^2 e^{2u_\epsilon} dv_g = \int_M |\Phi_\epsilon|^2 K_\epsilon[v_\epsilon] dv_g = \int_M K_\epsilon[|\Phi_\epsilon|^2] dv_\epsilon = 1$$

gives (4.17).

Let $\rho > 0$, $1 \leq i \leq n+1$ and up to change ϕ_ϵ^i into $-\phi_\epsilon^i$, let (x_ϵ) be a sequence of points such that $\phi_\epsilon^i(x_\epsilon) = \sup_{M(\rho)} |\phi_\epsilon^i|$. We set

$$\delta_\epsilon = d_g(x_\epsilon, \text{supp}(v_\epsilon)).$$

We divide the proof of (4.18) into three cases.

CASE 1 - We assume that $\delta_\epsilon^{-1} = O(1)$. Then, by (4.6), $\{e^{2u_\epsilon}\}$ is uniformly bounded in $B_g(x_\epsilon, \min\{\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\})$. By (4.17), $\{\phi_\epsilon^i\}$ is bounded in $L^2(M(\frac{\rho}{2}))$ and $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by standard elliptic theory on the eigenvalue equation.

CASE 2 - We assume that $\delta_\epsilon = O(\sqrt{\epsilon})$. Then, $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by Claim 19.

CASE 3 - We assume that $\delta_\epsilon \rightarrow 0$ and $\frac{\sqrt{\epsilon}}{\delta_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. We let

$$\psi_\epsilon = \tilde{\phi}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon) \text{ and } e^{2w_\epsilon} = \delta_\epsilon^2 e^{2\tilde{u}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon)}$$

for $x \in \mathbb{D}_{\delta\delta_\epsilon^{-1}}$ so that

$$\Delta\psi_\epsilon = \lambda_\epsilon e^{2w_\epsilon} \psi_\epsilon \text{ in } \mathbb{D}_{\delta\delta_\epsilon^{-1}}.$$

Let $y_\epsilon \in \text{supp}(\nu_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) = \delta_\epsilon$ and set $z_\epsilon = \frac{\tilde{y}_\epsilon - \tilde{x}_\epsilon}{\delta_\epsilon}$ so that $z_\epsilon \rightarrow z_0$ as $\epsilon \rightarrow 0$ up to the extraction of a subsequence. We set $R = |z_0|$. Thanks to Claim 19, we know that $\psi_\epsilon(z_\epsilon) = \phi_\epsilon^i(y_\epsilon) = O(1)$. Thanks to estimates (4.8) on the heat kernel, there exists $D_1 > 0$ such that

$$e^{2w_\epsilon} \leq D_1 \text{ on } \mathbb{D}_{\frac{R}{2}}.$$

We first assume that ψ_ϵ does not vanish in \mathbb{D}_{3R} . Then, we can apply Harnack's inequality and get some constant $D_2 > 0$ such that

$$\psi_\epsilon \geq D_2 \psi_\epsilon(0) \text{ on } \mathbb{D}_{\frac{R}{4}}$$

for all $\epsilon > 0$. Since ψ_ϵ is positive on $\mathbb{D}_{|z_\epsilon|}(z_\epsilon) \subset \mathbb{D}_{3R}$, by the equation, it is also superharmonic and we can write that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{2\pi|z_\epsilon|} \int_{\partial\mathbb{D}_{|z_\epsilon|}(z_\epsilon)} \psi_\epsilon d\sigma.$$

Taking only the part of the integral which lies in $\mathbb{D}_{\frac{R}{4}}$, we get the existence of some constant $D_3 > 0$ such that

$$\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0)$$

and this concludes the proof of (4.18) in this case since $\phi_\epsilon^i(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now assume that ψ_ϵ vanishes on \mathbb{D}_{3R} . Since $\delta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, and $x_\epsilon \in M(\rho)$, by Claim 20, ψ_ϵ vanishes on a piecewise smooth curve in \mathbb{D}_{4R} which connects two points of distance greater than R . By the corollary of Theorem 11 of Section 4.2.3 for $\Omega = \mathbb{D}_{5R}$, we get some constant $C_R > 0$ such that

$$\int_{\mathbb{D}_{4R}} \psi_\epsilon^2 \leq C_R \int_{\mathbb{D}_{5R}} |\nabla \psi_\epsilon|^2 dx$$

which proves that $\{\psi_\epsilon\}$ is bounded in $L^2(\mathbb{D}_{4R})$. By elliptic regularity, ψ_ϵ is bounded in $L^\infty(\mathbb{D}_{\frac{R}{4}})$ which gives that $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded. The study of these three cases completes the proof of (4.18).

Thanks to Claim 20, we have the existence of some $r_1(\rho) > 0$ such that for any $0 < r < r_1(\rho)$,

$$\forall \epsilon > 0, \forall x \in M(\rho), \frac{1}{\lambda_\star(B_g(x, r), e^{2u_\epsilon} g)} \leq \frac{2}{\Lambda_k(M, [g])}.$$

By isocapacity estimates (see [77], section 2.3.3, corollary of Theorem 2.3.2)

$$\begin{aligned} \int_{B_g(x,r)} e^{2u_\epsilon} dv_g &\leq \frac{\text{Cap}_2(B_g(x,r), B_g(x, r_1))}{\lambda_*(B_g(x,r), e^{2u_\epsilon} g)} \\ &\leq 2 \frac{\text{Cap}_2(\mathbb{D}_{\frac{r}{C_0}}, \mathbb{D}_{C_0 r_1})}{\Lambda_k(M, [g])} \\ &\leq \frac{4\pi}{\Lambda_k(M, [g]) \ln\left(\frac{C_0^2 r_1}{r}\right)} \end{aligned}$$

and we get (4.19).

Finally, following the proof of Claim 7 in [91], we can use (4.18) and (4.19), to get the estimate (4.20). This ends the proof of the claim. \diamond

In the following claim, we aim at passing to the limit in the equation (i) and the condition (ii) given by Proposition 2. The limiting functions would then satisfy (4.24) and (4.25).

Claim 22. *We assume that $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} e^{2u_\epsilon} dv_g > 0$ for any $\rho > 0$ small enough. Then, the following assertions hold :*

— For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x \in M(\rho), |\Phi_\epsilon|^2(x) \geq 1 - \beta_\epsilon. \quad (4.21)$$

— For $\rho > 0$ and $x \in M(\rho)$, we set $\Psi_\epsilon = \frac{\Phi_\epsilon}{|\Phi_\epsilon|}$. Then for any $\rho > 0$, $\{\Psi_\epsilon\}$ is uniformly equicontinuous on $C^0(M(\rho), \mathbb{S}^n)$.

— For any $\rho > 0$, up to the extraction of a subsequence of $\{\Phi_\epsilon\}$, there exist functions $\Phi \in W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \cap L^\infty(M(\rho), \mathbb{R}^{n+1})$ and $\Psi \in W^{1,2}(M(\rho), \mathbb{S}^n) \cap C^0(M(\rho), \mathbb{S}^n)$ such that

$$\Phi_\epsilon \rightharpoonup \Phi \text{ in } W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \text{ as } \epsilon \rightarrow 0 \quad (4.22)$$

and

$$\Psi_\epsilon \rightarrow \Psi \text{ in } C^0(M(\rho), \mathbb{S}^n) \text{ as } \epsilon \rightarrow 0 \quad (4.23)$$

with

$$|\Phi|^2 \geq_{a.e.} 1 \text{ and } \Psi = \frac{\Phi}{|\Phi|}. \quad (4.24)$$

Moreover, for $1 \leq i \leq n+1$

$$\Delta_g \phi^i = \Lambda_k(M, [g]) \psi^i dv \quad (4.25)$$

in a weak sense on $M(\rho)$.

Proof.

STEP 1 - Let $1 \leq i \leq s$. We prove that at the neighbourhood of the singular points defined in Claim 20,

$$\sup_{x \in M(\rho)} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) dv_g(y) = O(e^{-\frac{\rho^2}{8\epsilon}}).$$

Let $x \in M(\rho)$. Then, by estimates (4.5) of Section 4.2.2

$$\begin{aligned} e^{\frac{\rho^2}{8\epsilon}} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) dv_g(y) &\leq \frac{A_0}{4\pi\epsilon} e^{-\frac{31\rho^2}{400\epsilon}} \frac{\int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2 e^{2u_\epsilon} dv_g}{\inf_{B_g(p_i, \frac{\rho}{10})} e^{2u_\epsilon}} \\ &\leq \frac{A_0}{4\pi\epsilon} \frac{e^{-\frac{31\rho^2}{400\epsilon}}}{\inf_{B_g(p_i, \frac{\rho}{10})} e^{2u_\epsilon}} \end{aligned}$$

since by Proposition 2, (iii),

$$\int_M |\Phi_\epsilon|^2 e^{2u_\epsilon} dv_g = \int_M K_\epsilon[|\Phi_\epsilon|^2] dv_\epsilon = 1.$$

We assume by contradiction that

$$\inf_{B_g(p_i, \frac{\rho}{10})} e^{2u_\epsilon} \leq \frac{e^{-\frac{31\rho^2}{400\epsilon}}}{\epsilon}.$$

Let $y \in \overline{B_g(p_i, \frac{\rho}{10})}$ be such that $e^{2u_\epsilon(y)} = \inf_{B_g(p_i, \frac{\rho}{10})} e^{2u_\epsilon}$. Then, by (4.5) of Section 4.2.2,

$$e^{2u_\epsilon(y)} = \int_M p_\epsilon(y, x) dv_\epsilon(x) \geq \frac{e^{-\left(\frac{2\rho}{10}\right)^2 \frac{1}{4\epsilon}}}{4\pi A_0 \epsilon} \int_{B_g(p_i, \frac{\rho}{10})} dv_\epsilon.$$

We deduce from this and the previous inequality that

$$\int_{B_g(p_i, \frac{\rho}{10})} dv_\epsilon \leq 4\pi A_0 \epsilon e^{-\frac{27\rho^2}{400\epsilon}}.$$

Let $z \in B_g(p_i, \frac{\rho}{20})$, and let us write thanks again to (4.5) of Section 4.2.2 that

$$e^{2u_\epsilon(z)} \leq A_0 \frac{\int_{B_g(p_i, \frac{\rho}{10})} dv_\epsilon + e^{-\frac{\rho^2}{4\epsilon} \frac{1}{20^2}}}{4\pi\epsilon} \leq \frac{A_0^2}{\epsilon} e^{-\frac{27\rho^2}{400\epsilon}} + \frac{A_0}{4\pi\epsilon} e^{-\frac{\rho^2}{1600\epsilon}}.$$

Then, $\|e^{2u_\epsilon}\|_{C^0(B_g(p_i, \frac{\rho}{20}))} \rightarrow 0$ as $\epsilon \rightarrow 0$. Then $\lambda_\star(B_g(p_i, \frac{\rho}{20}), e^{2u_\epsilon} g) \rightarrow +\infty$ as $\epsilon \rightarrow 0$ which contradicts (4.13) in Claim 20. This completes the proof of Step 1.

STEP 2 - For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x \in M(\rho), \left| |\Phi_\epsilon|^2(x) - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon \quad (4.26)$$

and

$$\forall x \in M(\rho) \cap \text{supp}(v_\epsilon), |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon. \quad (4.27)$$

Note that (4.26) implies (4.21) by Proposition 2. We refer to the proof of Claim 8 in [91] to get $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x, y \in M(\rho), d_g(x, y) \leq \frac{\sqrt{\epsilon}}{\beta_\epsilon} \Rightarrow |\Phi_\epsilon(x) - \Phi_\epsilon(y)| \leq \beta_\epsilon \quad (4.28)$$

using (4.18), (4.19) and (4.20) of Claim 21. Let's prove (4.26). For $x \in M(\rho)$, we write thanks to (4.18) that

$$\begin{aligned} \left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right| (x) &\leq \int_{B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} \left| |\Phi_\epsilon|^2(x) - |\Phi_\epsilon|^2(y) \right| p_\epsilon(x, y) dv_g(y) \\ &\quad + 2C_2 \left(\frac{\rho}{10} \right)^2 \int_{M \setminus B_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x, y) dv_g(y) \\ &\quad + \sum_{i=1}^s \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2(y) p_\epsilon(x, y) dv_g(y). \end{aligned}$$

We can estimate the first right-hand side term thanks to (4.18) and (4.28), the second RHS term thanks to (4.5) of Section 4.2.2 and the third RHS term thanks to Step 1 to obtain that

$$\left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right| (x) \leq 2C_0(\rho)\beta_\epsilon + O(e^{-\frac{1}{4C_0^4\beta_\epsilon^2}}) + O(e^{-\frac{\rho^2}{8\epsilon}}).$$

Up to increase β_ϵ we get (4.26). Thanks to Proposition 2, (iii), we deduce, up to increase β_ϵ , that

$$\forall x \in M(\rho) \cap \text{supp}(\nu_\epsilon), ||\Phi_\epsilon(x)| - 1| \leq \beta_\epsilon.$$

By (4.26) and the previous arguments for $|\Phi_\epsilon|$ instead of $|\Phi_\epsilon|^2$, up to increase β_ϵ , we get (4.27).

Let $\rho > 0$. We follow Claim 9 of [91] in order to prove the uniform equicontinuity of the sequence Ψ_ϵ on $M(\rho)$, using the Poincaré inequality of Theorem 11 thanks to Claim 20. Therefore, up to the extraction of a subsequence we get some functions Φ and Ψ such that (4.22), (4.23) and (4.24) hold true. Let $1 \leq i \leq n+1$.

STEP 3 - We have that

$$\phi_\epsilon^i e^{2u_\epsilon} dv_g \rightharpoonup^* \psi^i dv \text{ as } \epsilon \rightarrow 0 \text{ in } \mathcal{M}(M(\rho)).$$

Let $\zeta \in \mathcal{C}_c^0(M(\rho))$. Then

$$\begin{aligned} \int_M \zeta \phi_\epsilon^i e^{2u_\epsilon} dv_g - \int_M \zeta \psi^i dv &= \int_M \left(K_\epsilon[\zeta \phi_\epsilon^i] - \zeta K_\epsilon[\phi_\epsilon^i] \right) dv_\epsilon \\ &\quad + \int_M \zeta \left(K_\epsilon[\phi_\epsilon^i] - \psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] \right) dv_\epsilon \\ &\quad + \int_M \zeta \left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] - \psi_\epsilon^i \right) dv_\epsilon \\ &\quad + \int_M \zeta \left(\psi_\epsilon^i dv_\epsilon - \psi^i dv \right). \end{aligned} \tag{4.29}$$

Let us estimate these four terms. We have for $x \in M$ that

$$\begin{aligned} \left| K_\epsilon[\zeta \phi_\epsilon^i] - \zeta K_\epsilon[\phi_\epsilon^i] \right| (x) &= \left| \int_M (\zeta(y) - \zeta(x)) \phi_\epsilon^i(y) p_\epsilon(x, y) dv_g(y) \right| \\ &\leq C_2 \left(\frac{\rho}{10} \right) \int_{M(\frac{\rho}{10})} |\zeta(y) - \zeta(x)| p_\epsilon(x, y) dv_g(y) \\ &\quad + |\zeta(x)| \sum_{j=1}^s \int_{B_g(p_j, \frac{\rho}{10})} |\phi_\epsilon^i(y)| p_\epsilon(x, y) dv_g(y) \end{aligned}$$

since $\text{supp}(\zeta) \subset M(\rho)$ and thanks to (4.18) of Claim 21. By Step 1 and since $\text{supp}(\zeta) \subset M(\rho)$, we deduce that this function uniformly converges to 0 in M as $\epsilon \rightarrow 0$. Thus, the first RHS term in (4.29) converges to 0 as $\epsilon \rightarrow 0$. For $x \in M(\rho)$,

$$\begin{aligned} \left| K_\epsilon[\phi_\epsilon^i] - \psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] \right| (x) &\leq \int_M \left| \phi_\epsilon^i(y) - \psi_\epsilon^i(x) |\phi_\epsilon|(y) \right| p_\epsilon(x, y) dv_g(y) \\ &\leq \int_{M(\frac{\rho}{10})} |\Phi_\epsilon(y)| \left| \psi_\epsilon^i(y) - \psi_\epsilon^i(x) \right| p_\epsilon(x, y) dv_g(y) \\ &\quad + 2 \sum_{j=1}^s \int_{B_g(p_j, \frac{\rho}{10})} |\Phi_\epsilon(y)| p_\epsilon(x, y) dv_g(y) \\ &\leq C_2 \left(\frac{\rho}{10} \right) \int_{M(\frac{\rho}{10})} \left| \psi_\epsilon^i(y) - \psi_\epsilon^i(x) \right| p_\epsilon(x, y) dv_g(y) \\ &\quad + O(e^{-\frac{\rho^2}{16\epsilon}}). \end{aligned}$$

thanks to (4.18) of Claim 21 and Step 1. Thanks to the uniform equicontinuity of $\{\Psi_\epsilon\}$ on $M(\frac{\rho}{10})$, it uniformly converges to zero in M as $\epsilon \rightarrow 0$. Thus, the second RHS term of (4.29) converges to 0 as $\epsilon \rightarrow 0$. Thanks to (4.27), we can write since $|\Psi_\epsilon| = 1$ that

$$\left| \int_M \zeta \left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] - \psi_\epsilon^i \right) dv_\epsilon \right| \leq \beta_\epsilon \|\zeta\|_\infty$$

so that the third RHS term in (4.29) converges to 0 as $\epsilon \rightarrow 0$. At last, we use the convergences $\Psi_\epsilon \rightarrow \Psi$ in $\mathcal{C}^0(M(\rho))$ and $\nu_\epsilon \rightharpoonup_\star \nu$ on $M(\rho)$ to obtain that the fourth RHS term in (4.29) also converges to 0 as $\epsilon \rightarrow 0$. This clearly ends the proof of Step 3.

Finally, passing to the weak limit in $M(\rho)$, for $\rho > 0$, in the equation satisfied by ϕ_ϵ^i permits to end the proof of the claim thanks to all these steps. \diamond

Thanks to Claim 22, with the assumption $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} e^{2u_\epsilon} dv_g > 0$, a diagonal extraction gives some functions $\Phi : M \setminus \{p_1, \dots, p_s\} \rightarrow \mathbb{R}^{n+1}$ and $\Psi : M \setminus \{p_1, \dots, p_s\} \rightarrow \mathbb{S}^n$ such that for all $\rho > 0$ the conclusions (4.22), (4.23), (4.24) and (4.25) hold true for Φ and Ψ .

4.4.2 Energy estimates

Now we give some energy estimates which will be useful later.

Claim 23.

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g \geq \int_M \frac{|\nabla \Phi|_g^2}{|\Phi|} dv_g \geq \Lambda_k(M, [g]) m + \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g \quad (4.30)$$

where $m = \lim_{\rho \rightarrow 0} m_0(\rho)$.

Proof.

By Claim 18, there exists $C > 0$ such that for any $\rho > 0$ there exists a nonnegative function $\eta \in \mathcal{C}^\infty(M)$ such that $\text{supp}(\eta) \subset M(\rho)$, $\eta = 1$ on $M(\sqrt{\rho})$, $0 \leq \eta \leq 1$ and

$$\int_M |\nabla \eta|_g^2 dv_g \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

Then, we test the equation (4.25) against $\eta\psi^i$ and sum over i to get that

$$\sum_{i=1}^{n+1} \int_M \langle \nabla \eta, \nabla \phi^i \rangle_g \psi^i dv_g + \sum_{i=1}^{n+1} \int_M \langle \nabla \psi^i, \nabla \phi^i \rangle_g \eta dv_g = \Lambda_k(M, [g]) \int_M \eta dv_g.$$

Since

$$\nabla \psi^i = \nabla \left(\frac{\phi^i}{|\Phi|} \right) = \frac{\nabla \phi^i}{|\Phi|} - \frac{\phi^i \nabla |\Phi|}{|\Phi|^2},$$

we deduce that

$$\Lambda_k(M, [g]) \int_M \eta dv_g = \int_M \langle \nabla \eta, \nabla |\Phi| \rangle_g dv_g + \int_M \left(\frac{|\nabla \Phi|_g^2}{|\Phi|} - \frac{|\nabla |\Phi||_g^2}{|\Phi|} \right) \eta dv_g.$$

Since $\Phi_\epsilon \rightharpoonup \Phi$ in $W^{1,2}(M(\rho), \mathbb{R}^{n+1})$ and $|\Phi| \geq_{a.e.} 1$, we can write that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g &\geq \int_{M(\rho)} |\nabla \Phi|_g^2 dv_g \\ &\geq \int_M \eta \frac{|\nabla \Phi|_g^2}{|\Phi|} dv_g \\ &\geq \Lambda_k(M, [g]) \int_M \eta dv_g - \int_M \langle \nabla \eta, \nabla |\Phi| \rangle_g dv_g \\ &\quad + \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} \eta dv_g \\ &\geq \Lambda_k(M, [g]) m_0(\sqrt{\rho}) - C' \sqrt{\frac{C}{\ln(\frac{1}{\rho})}} + \int_{M(\sqrt{\rho})} \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g \end{aligned}$$

where C and C' are some constants independent of ρ . Passing to the limit as $\rho \rightarrow 0$, we obtain the claim. \diamond

4.5 Scales of concentration for the maximizing sequence

4.5.1 Concentration, capacity and rescalings

In this section, we aim at describing all the concentration scales of the sequence $\{e^{2u_\epsilon} dv_g\}$. We denote by $Z(M, \{e^{2u_\epsilon} dv_g\})$ the concentration points of a sequence of measures $\{e^{2u_\epsilon} dv_g\}$ on a surface (M, g) , that is

$$Z(M, \{e^{2u_\epsilon} dv_g\}) = \{z \in M; \lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \int_{B_g(z, r)} e^{2u_\epsilon} dv_g > 0\}.$$

Taking the maximizing sequence $\{e^{2u_\epsilon} dv_g\}$ for $\lambda_k(M, [g])$ given by the previous section, which converges to ν in $\mathcal{M}_1(M)$, we clearly have that

$$Z(M, \{e^{2u_\epsilon} dv_g\}) = \{z \in M; \nu(\{z\}) > 0\}$$

and that

$$Z(M, \{e^{2u_\epsilon} g\}) \subset \bigcap_{r>0} X_r(M, \{e^{2u_\epsilon} g\}) = \{p_1, \dots, p_s\}, \quad (4.31)$$

where the p_i 's are given by Claim 20. This is a consequence of Claim 18 in Section 4.2.3 : indeed, for $x \in Z(M, \{e^{2u_\epsilon} dv_g\})$ and for $r > 0$ small enough, let η_{g,x,r,r^2} be given by Claim 18. Then

$$\lambda_\star(B_g(x, r), e^{2u_\epsilon}) \leq \frac{\int_M \left| \nabla \eta_{g,x,r,r^2} \right|_g^2 dv_g}{\int_M \left(\eta_{g,x,r,r^2} \right)^2 e^{2u_\epsilon} dv_g} \leq \frac{C}{\ln\left(\frac{1}{r}\right) \int_{B_g(x,r^2)} e^{2u_\epsilon} dv_g}$$

so that

$$\lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \lambda_\star(B_g(x, r), e^{2u_\epsilon}) = 0.$$

Then there is a subsequence $\{\epsilon_j\}$ for which x satisfies $\mathbf{A}_{r,\epsilon_j}$ for all r small enough. Thanks to Claim 20, this gives that $x \in \{p_1, \dots, p_s\}$.

We now define some functions which will rescale the problem at the neighbourhood of the concentration points. For $a \in \mathbb{R}^2$ and $\alpha > 0$, we let

$$H_{a,\alpha}(y) = \alpha y + a \text{ for } y \in \mathbb{R}^2.$$

For $p \in S^2$, we define the stereographic projection with respect to the pole p , $\sigma : S^2 \setminus \{p\} \rightarrow \mathbb{R}^2$, by

$$\sigma(z) = \frac{z - (z.p)p}{1 - (z.p)}$$

and its inverse

$$\sigma^{-1}(y) = \frac{2y - (1 - |y|^2)p}{1 + |y|^2}.$$

In this section, we prove the following :

Proposition 3. *There exist some points $a_1^\epsilon, \dots, a_N^\epsilon \in \mathbb{R}^2$ and some scales*

$$0 < \alpha_N^\epsilon < \alpha_{N-1}^\epsilon < \dots < \alpha_1^\epsilon$$

such that for $1 \leq i \leq N$,

$$\alpha_i^\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (4.32)$$

and letting

$$F_i = \left\{ j > i; \frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon} \text{ is bounded} \right\},$$

we have for $j \neq i$ that

$$j \in F_i \Rightarrow \frac{\alpha_j^\epsilon}{\alpha_i^\epsilon} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (4.33)$$

and that

$$j \notin F_i \Rightarrow \frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0. \quad (4.34)$$

There exist some disjoint sets $M_0^\epsilon, M_1^\epsilon, \dots, M_N^\epsilon \subset M$, some sets $D_1^\epsilon, \dots, D_N^\epsilon \subset \mathbb{R}^2$ and $S_1^\epsilon, \dots, S_N^\epsilon \subset \mathbb{S}^2$ given by

$$D_i^\epsilon = H_{a_i^\epsilon, \alpha_i^\epsilon}^{-1} \left(\widetilde{M}_i^{\epsilon l_i} \right) \text{ and } S_i^\epsilon = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \circ \sigma \right)^{-1} \left(\widetilde{M}_i^{\epsilon l_i} \right),$$

some associated densities defined by

$$e^{2\tilde{u}_i^\epsilon} d\tilde{\zeta} = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \right)^* \left(e^{2\tilde{u}_i^{l_i}} d\tilde{\zeta} \right) \text{ and } e^{2\tilde{u}_i^\epsilon} dv_h = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \circ \sigma \right)^* \left(e^{2\tilde{u}_i^{l_i}} d\tilde{\zeta} \right),$$

some masses $m_i > 0$ satisfying

$$\text{Vol}_{e^{2u_\epsilon} g}(M_i^\epsilon) = \text{Vol}_{e^{2\tilde{u}_i^\epsilon} \tilde{\zeta}}(D_i^\epsilon) = \text{Vol}_{e^{2\tilde{u}_i^\epsilon} h}(S_i^\epsilon) \rightarrow m_i \text{ as } \epsilon \rightarrow 0 \quad (4.35)$$

for $1 \leq i \leq N$ and some $l_i \in \{1, \dots, L\}$, and $m_0 \geq 0$ satisfying

$$\text{Vol}_{e^{2u_\epsilon} g}(M_0^\epsilon) \rightarrow m_0 \text{ as } \epsilon \rightarrow 0 \quad (4.36)$$

such that

$$Z(\mathbb{S}^2, \{\mathbf{1}_{S_i^\epsilon} e^{2\tilde{u}_i^\epsilon} dv_h\}) = \emptyset \quad (4.37)$$

for $1 \leq i \leq N$,

$$Z(M, \{\mathbf{1}_{M_0^\epsilon} e^{2u_\epsilon} dv_g\}) = \emptyset \quad (4.38)$$

and

$$\sum_{i=0}^N m_i = 1. \quad (4.39)$$

4.5.2 Proof of Proposition 3

Let us denote by z_1, \dots, z_{N_0} the atoms of ν with $N_0 \leq s \leq k$, where s is given by Claim 20 (see (4.31)) so that

$$e^{2u_\epsilon} dv_g \rightharpoonup^* \nu_0 + \sum_{i=1}^{N_0} m_i \delta_{z_i}$$

where $\nu_0 \in \mathcal{M}(M)$ has no atoms. Let $m_0 = \int_M d\nu_0 \geq 0$. All the m_i 's are positive for $1 \leq i \leq N_0$, and

$$\sum_{i=0}^{N_0} m_i = 1.$$

Let $1 \leq i \leq N_0$. We choose $l_i \in \{1, \dots, L\}$ such that $z_i \in \omega_{l_i}$. Up to the extraction of a subsequence, one can build a sequence $\{r_i^\epsilon\}$ such that $r_i^\epsilon > 0$ and $r_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ with

$$\int_{B_g(z_i, r_i^\epsilon)} e^{2u_\epsilon} dv_g \rightarrow m_i \text{ as } \epsilon \rightarrow 0.$$

We associate to sequences $a_i^\epsilon \in \mathbb{R}^2$ and $\alpha_i^\epsilon > 0$ that we shall choose later the sets

$$D_i^\epsilon = H_{a_i^\epsilon, \alpha_i^\epsilon}^{-1} \left(\widetilde{B_g(z_i, r_i^\epsilon)}^{l_i} \right) \subset \mathbb{R}^2,$$

$$S_i^\epsilon = \sigma^{-1}(D_i^\epsilon) \subset \mathbb{S}^2,$$

$$M_i^\epsilon = B_g(z_i^\epsilon, r_i^\epsilon) \text{ and}$$

$$M_0^\epsilon = M \setminus \bigcup_{i=1}^{N_0} M_i^\epsilon$$

and the densities

$$e^{2\hat{u}_i^\epsilon} = (\alpha_i^\epsilon)^2 e^{2(\hat{u}_i^\epsilon + \tilde{v}_{l_i^\epsilon}^\epsilon) \circ H_{a_i^\epsilon, \alpha_i^\epsilon}} : D_i^\epsilon \rightarrow \mathbb{R} \text{ and}$$

$$e^{2\hat{u}_i^\epsilon} dv_h = \sigma^*(e^{2\hat{u}_i^\epsilon} dx) : S_i^\epsilon \rightarrow \mathbb{R}.$$

For the notations, we refer to Section 4.2.1.

Note that

$$M = M_0^\epsilon \cup \bigcup_{i=1}^{N_0} M_i^\epsilon$$

with $\text{Vol}_{e^{2u_\epsilon} g}(M_i^\epsilon) \rightarrow m_i$ as $\epsilon \rightarrow 0$ for $0 \leq i \leq N_0$. We assign to the subset M_i^ϵ a test function $\eta_i^\epsilon \in C_c^\infty(M_i^\epsilon)$ given by Claim 18 in Section 4.2.3

$$\eta_i^\epsilon = \eta_{g, z_i, (r_i^\epsilon)^{\frac{1}{2}}, r_i^\epsilon} \text{ for } 1 \leq i \leq N_0,$$

$$\eta_0^\epsilon = 1 - \sum_{i=1}^{N_0} \eta_{g, z_i, (r_i^\epsilon)^{\frac{1}{4}}, (r_i^\epsilon)^{\frac{1}{2}}}.$$

Note that these test functions with pairwise disjoint supports and small Rayleigh quotient may also be used to prove that $N_0 \leq k$ if $m_0 = 0$ or $N_0 \leq k - 1$ if $m_0 > 0$.

For $1 \leq i \leq N_0$, let's now adjust the parameters a_i^ϵ and α_i^ϵ in order to detect other scales of concentration of the mass in the neighbourhood of z_i . By Hersch theorem (see [54], lemma 1.1), we can choose $a_i^\epsilon \in \mathbb{R}^2$ and $\alpha_i^\epsilon > 0$ such that

$$\int_{S^2} x e^{2\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} dv_h = 0. \quad (4.40)$$

Note that $\bar{a}_i^\epsilon \rightarrow z_i$ and that $\alpha_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. This normalization of the center of mass gives a dichotomy in the description of the concentration points of $\{e^{2\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} dv_h\}$: if $z \in Z(S^2, \{e^{2\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} dv_h\})$, then some mass is also concentrated in the opposite hemisphere $\{x \in S^2; (x, z) \leq 0\}$ and we can increase the number of test functions with small Rayleigh quotient on the manifold among $\eta_1^\epsilon, \dots, \eta_{N_0}^\epsilon$. From this remark, we will build by induction a finite bubble tree which describes the concentrations at all the scales they appear.

A tree T is a set of finite sequences

$$\gamma = (i_1, \dots, i_{|\gamma|}) \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$$

where $|\gamma|$ is the length of γ which satisfies

- $(\emptyset) \in T$ is the root of the tree
- if $\gamma \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$ and $i \in \mathbb{N}$, then $(\gamma, i) \in T \Rightarrow \gamma \in T$ and (γ, i) is called a son of γ .
- If $(\gamma, 0) \in T$ then $\forall i \in \mathbb{N}, (\gamma, 0, i) \notin T$. $(\gamma, 0)$ is called a leaf of T . We denote by L_T the set of leaves of T .

- If $\gamma \in T$, then $\{i \in \mathbb{N}; (\gamma, i) \in T\} = \{0, \dots, N_\gamma\}$ with $N_\gamma \in \mathbb{N}$ and N_γ is the number of sons of γ .

Let T be a tree. We let $|T| = \sup\{|\gamma|; \gamma \in T\}$ be the depth of the tree. We let also $T_j = \{\gamma \in T; |\gamma| \leq j\}$ be the truncated tree of depth $j \in \mathbb{N}$. We say that $\tilde{\gamma} \in T$ is a descendant of $\gamma \in T$ if there exists $\gamma' \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$ such that $\tilde{\gamma} = (\gamma, \gamma')$.

In the following, we define by induction a tree T with

- some sets $M_\gamma^\epsilon \subset M$ for $\gamma \in T$ and $D_\gamma^\epsilon \subset \mathbb{R}^2$, $S_\gamma^\epsilon \subset \mathbb{S}^2$ for $\gamma \in T \setminus L_T$,
- some parameters $l_\gamma \in \{1, \dots, L\}$, $r_\gamma^\epsilon > 0$, $a_\gamma^\epsilon \in \mathbb{R}^2$ and $\alpha_\gamma^\epsilon > 0$ for $\gamma \in T \setminus L_T$,
- some points $z_\gamma \in \mathbb{S}^2$ if $\gamma \in T \setminus L_T$ and $|\gamma| \geq 2$ and $z_\gamma \in M$ if $\gamma \in T \setminus L_T$ and $|\gamma| = 1$,
- some measures $\nu_0 \in \mathcal{M}(M)$ of mass $m_0 = \int_M d\nu_0 \geq 0$, $\nu_\gamma \in \mathcal{M}(\mathbb{S}^2)$ of mass $m_\gamma = \int_{\mathbb{S}^2} d\nu_\gamma \geq 0$ if $\gamma \in L_T$ and $|\gamma| \geq 2$ and some masses $m_\gamma > 0$ for $\gamma \in T \setminus L_T$,
- some functions $\hat{u}_\gamma^\epsilon : D_\gamma^\epsilon \rightarrow \mathbb{R}$ and $\check{u}_\gamma^\epsilon : S_\gamma^\epsilon \rightarrow \mathbb{R}$,
- some test functions $\eta_\gamma^\epsilon : M \rightarrow \mathbb{R}$ with $\eta_\gamma^\epsilon \in \mathcal{C}_c^\infty(M_\gamma^\epsilon)$ for $\gamma \in T$

depending on ϵ . We describe the process of construction, by induction, of this tree now and will prove in Claim 24 that it is a finite tree.

If $\gamma \in T$ and $|\gamma| = 1$, these objects are defined at the beginning of Section 4.5.2.

Assume now that these objects are defined for all γ of length $|\gamma| \leq j$. Let $\gamma \in T \setminus L_T$ with $|\gamma| \leq j$. Then, up to the extraction of a subsequence,

$$\mathbf{1}_{S_\gamma^\epsilon} e^{2\check{u}_\gamma^\epsilon} d\nu_h \rightharpoonup^* \nu_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}} \quad (4.41)$$

where for $1 \leq i \leq N_\gamma$, $m_{(\gamma,i)} > 0$, $m_{(\gamma,0)} = \int_{\mathbb{S}^2} d\nu_{(\gamma,0)}$ and $\nu_{(\gamma,0)}$ is without atom. As we will see in the proof of Claim 24 and by the same arguments as in the previous subsection, Claim 18 provides some test functions which prove that $N_\gamma \leq k$. Notice that

$$\sum_{i=0}^{N_\gamma} m_{(\gamma,i)} = m_\gamma.$$

Let $1 \leq i \leq N_\gamma$. We define $l_{(\gamma,i)} = l_\gamma$ and up to the extraction of a subsequence, we can build $\{r_{(\gamma,i)}^\epsilon\}$ such that $r_{(\gamma,i)}^\epsilon > 0$ and $r_{(\gamma,i)}^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ with

$$\int_{B_h(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon) \cap S_\gamma^\epsilon} e^{2\check{u}_\gamma^\epsilon} d\nu_h \rightarrow m_{(\gamma,i)} \text{ as } \epsilon \rightarrow 0.$$

We define

$$\bar{\eta}_{(\gamma,i)}^\epsilon = \eta_{h, z_{(\gamma,i)}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}, r_{(\gamma,i)}^\epsilon} \circ \sigma^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{gl_\gamma, x_{l_\gamma}}^{-1}$$

and

$$\bar{\eta}_{(\gamma,0)}^\epsilon = 1 - \sum_{i=1}^{N_\gamma} \eta_{h, z_{(\gamma,i)}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{4}}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}} \circ \sigma^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{gl_\gamma, x_{l_\gamma}}^{-1}$$

naturally extended by a constant on M so that $\bar{\eta}_{(\gamma,i)}^\epsilon \in \mathcal{C}^\infty(M)$. For $0 \leq i \leq N_\gamma$ the function

$$\eta_{(\gamma,i)}^\epsilon = \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon$$

satisfies (4.43) in the proof of Claim 24 and that

$$\text{supp}(\eta_{(\gamma,i)}^\epsilon) \cap \text{supp}(\eta_{(\gamma,j)}^\epsilon) = \emptyset \text{ for } i \neq j \text{ and } \text{supp}(\eta_{(\gamma,i)}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon).$$

The use of these test functions proves that $N_\gamma \leq k$.

Let $1 \leq i \leq N_\gamma$. We define the sets

$$D_{(\gamma,i)}^\epsilon = H_{a_{(\gamma,i)}^\epsilon, \alpha_{(\gamma,i)}^\epsilon}^{-1} \left(H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon} \left(D_\gamma^\epsilon \cap \sigma^{-1} \left(B_h(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon) \right) \right) \right),$$

$$S_{(\gamma,i)}^\epsilon = \sigma^{-1} \left(D_{(\gamma,i)}^\epsilon \right),$$

$$M_{(\gamma,i)}^\epsilon = \exp_{g_{l_\gamma}, x_{l_\gamma}} \left(H_{a_{(\gamma,i)}^\epsilon, \alpha_{(\gamma,i)}^\epsilon} \left(D_{(\gamma,i)}^\epsilon \right) \right) = \check{D}_{(\gamma,i)}^\epsilon,$$

$$M_{(\gamma,0)}^\epsilon = M_\gamma^\epsilon \setminus \bigcup_{i=1}^{N_\gamma} M_{(\gamma,i)}^\epsilon$$

and the densities

$$\frac{e^{2\hat{u}_{(\gamma,i)}^\epsilon \left(\frac{z - a_{(\gamma,i)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \right)}}{\left(\alpha_{(\gamma,i)}^\epsilon \right)^2} = \frac{e^{2\hat{u}_\gamma^\epsilon \left(\frac{z - a_\gamma^\epsilon}{\alpha_\gamma^\epsilon} \right)}}{\left(\alpha_\gamma^\epsilon \right)^2},$$

$$e^{2\check{u}_{(\gamma,i)}^\epsilon} dv_h = \sigma^* \left(e^{2\hat{u}_{(\gamma,i)}^\epsilon} dx \right),$$

and by Hersch's normalization, we choose the parameters $a_{(\gamma,i)}^\epsilon$ and $\alpha_{(\gamma,i)}^\epsilon$ with

$$\int_{S^2} x e^{2\check{u}_{(\gamma,i)}^\epsilon} \mathbf{1}_{S_{(\gamma,i)}^\epsilon} dv_h = 0 \quad (4.42)$$

and

$$\int_{M_{(\gamma,i)}^\epsilon} e^{2u_\epsilon} dv_g = \int_{D_{(\gamma,i)}^\epsilon} e^{2\hat{u}_{(\gamma,i)}^\epsilon} = \int_{S_{(\gamma,i)}^\epsilon} e^{2\check{u}_{(\gamma,i)}^\epsilon} = m_{(\gamma,i)}.$$

Claim 24. T is a finite tree.

Proof.

STEP 1 - We prove that if $\gamma \in T \setminus L_T$, then

$$\text{either } N_\gamma = 0 \text{ or } \#\{0 \leq i \leq N_\gamma; m_{(\gamma,i)} > 0\} \geq 2$$

Since $m_{(\gamma,i)} > 0$ for $1 \leq i \leq N_\gamma$, we get Step 1 if $N_\gamma \geq 2$ or $N_\gamma = 0$. We now assume that $N_\gamma = 1$. By (4.41) and (4.42),

$$\int_{S^2} (x, z_{(\gamma,1)}) dv_{(\gamma,0)} + m_{(\gamma,1)} = 0$$

Since $m_{(\gamma,1)} > 0$, we get that $v_{(\gamma,0)} \neq 0$ and $m_{(\gamma,0)} > 0$. This proves Step 1.

STEP 2 - We prove that if $\gamma \in T \setminus L_T$, then

$$\frac{\int_M \left| \nabla \eta_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_M \left(\eta_{(\gamma,i)}^\epsilon \right)^2 e^{2u_\epsilon} dv_g} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (4.43)$$

and that if $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}|$, then

- if $\tilde{\gamma}$ is not a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \cap \text{supp}(\eta_\gamma^\epsilon) = \emptyset$
- if $\tilde{\gamma}$ is a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon) = \emptyset$

We prove (4.43) by induction on $|\gamma| \leq j$. This is clearly true for $j = 1$. We assume it is true for all $|\gamma| \leq j$ with $j \geq 1$. We have

$$\frac{\int_M \left| \nabla \eta_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_M \left(\eta_{(\gamma,i)}^\epsilon \right)^2 e^{2u_\epsilon} dv_g} = \frac{\int_M \left| \nabla \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_M \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right)^2 e^{2u_\epsilon} dv_g}$$

with

$$\begin{aligned} \int_M \left| \nabla \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g &\leq 2 \left(\int_M \left| \nabla \eta_\gamma^\epsilon \right|_g^2 dv_g + \int_M \left| \nabla \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g \right) \\ &= 2 \left(o \left(\int_M (\eta_\gamma^\epsilon)^2 e^{2u_\epsilon} dv_g \right) + o(1) \right) \end{aligned}$$

by the induction assumption and for $i \geq 1$,

$$\begin{aligned} \int_M \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right)^2 e^{2u_\epsilon} dv_g &\geq \int_{\mathbb{S}^2} \left(\eta_{h,z_{(\gamma,i)},(r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}},(r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}} \right)^2 e^{2\tilde{u}_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} dv_h \\ &\geq \int_{\mathbb{S}^2} e^{2\tilde{u}_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon \cap B_h(z_{(\gamma,i)},(r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}})} dv_h \\ &= m_{(\gamma,i)} \end{aligned}$$

and for $i = 0$, fixing $\rho > 0$,

$$\begin{aligned} \int_M \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,0)}^\epsilon \right)^2 e^{2u_\epsilon} dv_g &\geq \int_{\mathbb{S}^2} \left(1 - \sum_{i=1}^{N_\gamma} \eta_{h,z_{(\gamma,i)},(r_{(\gamma,i)}^\epsilon)^{\frac{1}{4}},(r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}} \right)^2 e^{2\tilde{u}_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} dv_h \\ &\geq \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^{N_\gamma} B_h(p_i, \rho)} e^{2\tilde{u}_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} dv_h \\ &= \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^{N_\gamma} B_h(p_i, \rho)} \left(dv_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}} \right) + o(1) \\ &= \int_{\mathbb{S}^2 \setminus B_h(p_i, \rho)} dv_{(\gamma,0)} + o(1) \end{aligned}$$

as $\epsilon \rightarrow 0$. Gathering the previous inequalities, together with

$$\int_M (\eta_\gamma^\epsilon)^2 e^{2u_\epsilon} dv_g \leq \int_M e^{2u_\epsilon} dv_g = 1$$

we get (4.43).

We now prove the second part of step 2, also by induction. Assume that, for some $j \geq 1$ fixed, for all $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j$ we have that

- If $\tilde{\gamma}$ is not a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \cap \text{supp}(\eta_\gamma^\epsilon) = \emptyset$.
- If $\tilde{\gamma}$ is a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon)$.

Let us prove now that this is still true for any $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j+1$. If $|\tilde{\gamma}| \leq j$, there is of course nothing to prove. Assume that $|\tilde{\gamma}| = j+1$

If $|\gamma| = j+1$, then,

$$\text{supp}(\eta_\gamma^\epsilon) \cap \text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\bar{\eta}_\gamma^\epsilon) \cap \text{supp}(\bar{\eta}_{\tilde{\gamma}}^\epsilon)$$

which is empty if and only if $\gamma \neq \tilde{\gamma}$.

If $|\gamma| \leq j$, we denote $\tilde{\gamma} = (\hat{\gamma}, i)$ with $0 \leq i \leq N_{\hat{\gamma}}$. We can apply the induction hypothesis to $|\gamma| \leq |\hat{\gamma}| \leq j$. Then,

- if $\text{supp}(\eta_{\tilde{\gamma}}) \cap \text{supp}(\eta_\gamma) \neq \emptyset$, we get $\text{supp}(\eta_{\hat{\gamma}}) \cap \text{supp}(\eta_\gamma) \neq \emptyset$ since $\text{supp}(\eta_{\tilde{\gamma}}) \subset \text{supp}(\eta_{\hat{\gamma}})$. By the induction assumption, $\hat{\gamma}$ is a descendant of γ and $\tilde{\gamma}$ is a descendant of γ .
- If $\tilde{\gamma}$ is a descendant of γ , then, $\hat{\gamma}$ is a descendant of γ and by the induction assumption, $\text{supp}(\eta_{\tilde{\gamma}}) \subset \text{supp}(\eta_{\hat{\gamma}}) \subset \text{supp}(\eta_\gamma)$.

The proof of Step 2 is complete.

STEP 3 - We prove the following assertion \mathbf{H}_j by induction on j .

\mathbf{H}_j : If $T_j \neq T_{j+1}$, then, $T_{j+1} = T$ or there exist $j+1$ test functions with pairwise disjoint support in the set $\{\eta_\gamma^\epsilon, \gamma \in T_{j+1}\}$.

Notice that by (4.43) in Step 2, the assumption $T_{k+1} \neq T$ would give a contradiction. Indeed, it suffices to test the $k+1$ functions given by the assumption \mathbf{H}_{k+1} in the variational characterization (4.2.1) of $\lambda_\epsilon = \lambda_k(M, e^{2u_\epsilon}g)$. Therefore, the increasing sequence of trees $\{T_j\}$ is stationnary, and Claim 24 will follow.

Note that \mathbf{H}_1 is true by the existence of $\{\eta_1^\epsilon\}$.

Let $j \geq 1$ and we assume that \mathbf{H}_{j-1} is true and that $T_j \neq T_{j+1}$. Then, $T_{j-1} \neq T_j$ and \mathbf{H}_{j-1} gives j test functions with pairwise disjoint support in the set $\{\eta_\gamma^\epsilon; \gamma \in T_j\}$ denoted by $\eta_{\gamma_1}^\epsilon \cdots \eta_{\gamma_j}^\epsilon$. We assume that $T_{j+1} \neq T$. Then, there is $\gamma \in T_j$ such that $N_\gamma \geq 1$. By Step 1, there are two indices $i_1 \neq i_2$ such that $m_{(\gamma, i_1)} > 0$ and $m_{(\gamma, i_2)} > 0$.

If γ is not a descendant of one of $\gamma_1, \dots, \gamma_j$, then we take the set of test functions

$$\{\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon, \eta_{(\gamma, i_1)}^\epsilon\}.$$

If γ is a descendant of one of $\gamma_1, \dots, \gamma_j$, then, by Step 2, since the functions $\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon$ have pairwise disjoint support, there is a unique $1 \leq i \leq j$ such that γ is a descendant of γ_i and we take the set of test functions with pairwise disjoint support

$$\{\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_{i-1}}^\epsilon, \eta_{\gamma_{i+1}}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon, \eta_{(\gamma, i_1)}^\epsilon, \eta_{(\gamma, i_2)}^\epsilon\}.$$

Thus \mathbf{H}_j holds. This ends the proof of Step 3 and as already said the proof of Claim 24.



Thanks to this construction, the parameters $(a_\gamma^\epsilon, \alpha_\gamma^\epsilon)$ define separated bubble or bubbles over bubbles. This reads as a formula which originates from [8] and [106] in the context of bubble tree constructions :

Claim 25. *If $\gamma \in T \setminus L_T$, $\alpha_\gamma^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and if $\gamma_1, \gamma_2 \in T \setminus L_T$ with $\gamma_1 \neq \gamma_2$, then*

$$\frac{d_g(\bar{a}_{\gamma_1}^\epsilon, \bar{a}_{\gamma_2}^\epsilon)}{\alpha_{\gamma_1}^\epsilon + \alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_1}^\epsilon}{\alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_2}^\epsilon}{\alpha_{\gamma_1}^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Proof. We recall that there exists $C_0 > 0$ such that for all $0 < r < \delta$,

$$B_g(x, C_0^{-1}r) \subset \exp_{g_l, x_l}(\mathbb{D}_r(\tilde{x}^l)) \subset B_g(x, C_0 r)$$

for all $x \in \omega_l$ with $1 \leq l \leq L$. On the spheres, there also exists $C_1 > 0$ and some $\delta_1 > 0$ such that for all $0 < r < \delta_1$,

$$B_h(z_\gamma, C_1^{-1}r) \subset \sigma^{-1}(\mathbb{D}_r(\hat{z}_\gamma)) \subset B_h(z_\gamma, C_1 r)$$

for all $\gamma \in T \setminus L_T$ such that $|\gamma| \geq 2$ and $z_\gamma \neq p$, where $\hat{z}_\gamma = \sigma(z_\gamma)$; and

$$B_h(p, C_1^{-1}r) \subset \sigma^{-1}(\mathbb{R}^2 \setminus \mathbb{D}_{\frac{1}{r}}) \subset B_h(p, C_1 r).$$

Now, given $\gamma_1, \gamma_2 \in T \setminus L_T$, we let $\gamma \in T$ such that $\gamma_1 = (\gamma, \tilde{\gamma}_1)$, $\gamma_2 = (\gamma, \tilde{\gamma}_2)$ and $|\gamma|$ is maximal. We consider 5 cases in order to prove the claim.

CASE 1 - $\gamma = (\emptyset)$. Then $\gamma_1 = (i, \hat{\gamma}_1)$ and $\gamma_2 = (j, \hat{\gamma}_1)$ with $i \neq j$.

Since

$$M_{\gamma_1}^\epsilon \subset B_g(z_i, r_i^\epsilon) \subset \exp_{g_l, x_l}(\mathbb{D}_{C_0 r_i^\epsilon}(\tilde{z}_i)),$$

we get with (4.40) that

$$|a_i^\epsilon - \tilde{z}_i| \leq C_0 r_i^\epsilon$$

and

$$\alpha_i^\epsilon \leq C_0 r_i^\epsilon + |a_i^\epsilon - \tilde{z}_i|$$

so that $a_i^\epsilon \rightarrow \tilde{z}_i$ as $\epsilon \rightarrow 0$ and $\alpha_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and the same is true for j . Then, since $z_i \neq z_j$,

$$\frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon + \alpha_j^\epsilon} = \frac{d_g(z_i, z_j) + o(1)}{\alpha_i^\epsilon + \alpha_j^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

CASE 2 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma, j)} \neq p$.

Then, we have

$$M_{\gamma_2}^\epsilon \subset M_{(\gamma, j)}^\epsilon \subset \exp_{g_l, x_l}(\mathbb{D}_{C_1 r_{(\gamma, j)}^\epsilon}(\alpha_\gamma^\epsilon \hat{z}_{(\gamma, j)} + a_\gamma^\epsilon))$$

so that by (4.42), we have that

$$|\alpha_\gamma^\epsilon \hat{z}_{(\gamma, j)} + a_\gamma^\epsilon - a_{(\gamma, j)}^\epsilon| \leq C_1 r_{(\gamma, j)}^\epsilon \alpha_\gamma^\epsilon$$

and

$$\alpha_{(\gamma,j)}^\epsilon \leq C_1 r_{(\gamma,j)}^\epsilon \alpha_\gamma^\epsilon + \left| \alpha_\gamma^\epsilon \hat{z}_{(\gamma,j)} + a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon \right|$$

and $\frac{\alpha_\gamma^\epsilon}{\alpha_{(\gamma,j)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

CASE 3 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,j)} = p$.

We assume that $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_{(\gamma,j)}^\epsilon + \alpha_\gamma^\epsilon}$ is bounded and we prove by contradiction that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. We assume that $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_\gamma^\epsilon)$. Then, it is clear that $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon}$ is bounded and we have by (4.42) that

$$\alpha_{(\gamma,j)}^\epsilon \geq \frac{\alpha_\gamma^\epsilon}{C_1 r_{(\gamma,i)}^\epsilon} - \left| a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon \right|$$

so that

$$\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \geq \frac{1}{C_1 r_{(\gamma,i)}^\epsilon} - \frac{|a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon|}{\alpha_\gamma^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

which contradicts the assumption $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_\gamma^\epsilon)$. Thus, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

CASE 4 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $i \neq j$, $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} \neq p$.

We have that $|a_{(\gamma,i)}^\epsilon - a_{(\gamma,j)}^\epsilon| = \alpha_\gamma^\epsilon \left(|\hat{z}_{(\gamma,i)} - \hat{z}_{(\gamma,j)}| + o(1) \right)$, $\frac{\alpha_{(\gamma,i)}^\epsilon}{\alpha_\gamma^\epsilon} = o(1)$ and $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} = o(1)$ by Case 2 so that

$$\frac{d_g(\bar{a}_{(\gamma,i)}^\epsilon, \bar{a}_{(\gamma,j)}^\epsilon)}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

CASE 5 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} = p$.

As in Case 3, we assume that $\frac{|a_{(\gamma,i)}^\epsilon - a_{(\gamma,j)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon}$ is bounded and we will prove by contradiction that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. Let's assume that $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_{(\gamma,i)}^\epsilon)$. Then,

$$\begin{aligned} \frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + \frac{|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} \\ &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + \frac{|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + o(\alpha_\gamma^\epsilon)} \\ &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + O(1) \end{aligned}$$

since $\alpha_{(\gamma,i)}^\epsilon = o(\alpha_\gamma^\epsilon)$ by Case 2, and $|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon| = O(\alpha_\gamma^\epsilon)$. Then, $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon}$ is bounded and by Case 3, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ so that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ which gives a contradiction. Thus, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

Gathering all the cases, the proof is complete. \diamond

Now, we are in position to prove Proposition 3. We denote by $L^+ \subset L_T$ the set of leaves $\gamma \in L_T$ such that $m_\gamma > 0$.

To simplify, we now denote the elements of L^+ by $\{1, \dots, N\}$ and all the indices $\gamma \in L^+$ in $M_\gamma^\epsilon, D_\gamma^\epsilon, S_\gamma^\epsilon, a_\gamma^\epsilon, \alpha_\gamma^\epsilon, e^{2\hat{u}_\gamma^\epsilon}, e^{2\check{u}_\gamma^\epsilon}, v_\gamma$ and m_γ are replaced by the corresponding index $i \in \{1, \dots, N\}$.

Up to the extraction of a subsequence and up to reorder the $\alpha_i^{\epsilon'}$ s, we get (4.32), (4.33) and (4.34) thanks to Claim 25. By construction, we obtain the remaining facts of the proposition.

4.6 Regularity estimates at the concentration scales

In this section, we aim at proving some energy estimates in order to prove later Proposition 4 page 142. We fix $i \in \{1, \dots, N\}$ given by Proposition 3 and up to the end of the section drop the index i of the parameters $l_i, a_i^\epsilon, \alpha_i^\epsilon$ the functions \hat{u}_i^ϵ , we defined. As described in Section 4.2.1, we let

$$\hat{\Phi}_\epsilon(z) = \tilde{\Phi}_\epsilon^l \circ H_{a_\epsilon, \alpha_\epsilon}(z) = \tilde{\Phi}_\epsilon^l(\alpha_\epsilon z + a_\epsilon)$$

and

$$\hat{v}_\epsilon = H_{a_\epsilon, \alpha_\epsilon}^*(\tilde{v}_\epsilon).$$

Then, for $1 \leq i \leq n+1$ and for $\rho > 0$ fixed, we get the equations

$$\Delta_{\tilde{\zeta}} \hat{\Phi}_\epsilon^i = \lambda_\epsilon e^{2\hat{u}_\epsilon} \hat{\Phi}_\epsilon^i \text{ on } \mathbb{D}_{\frac{1}{\rho}}. \quad (4.44)$$

As we will see, the properties gathered in Proposition 2 and Claim 20 are in some sense invariant by dilatation. Indeed, this is clear in the equation (4.44). We also have that if $\Omega \subset \omega_l$,

$$\lambda_\star(\Omega, e^{2\hat{u}_\epsilon}) = \lambda_\star(\hat{\Omega}, e^{2\hat{u}_\epsilon})$$

where we set $\hat{\Omega} = H_{a_\epsilon, \alpha_\epsilon}^{-1}(\tilde{\Omega}^l)$. The heat equation is also invariant by dilatation, up to some errors on the surface M we precised in Section 4.2.2 (see (4.4) and (4.6)), thanks to the following identity in the Euclidean case

$$\int_{\mathbb{R}^2} \frac{1}{4\pi\epsilon} e^{-\frac{|x-y|^2}{4\epsilon}} f(y) dy = \int_{\mathbb{R}^2} \frac{\alpha^2}{4\pi\epsilon} e^{-\alpha^2 \frac{|x-y|^2}{4\epsilon}} f(\alpha y) dy.$$

Therefore, we can derive regularity estimates of the eigenfunctions at all the concentration scales.

However, we have to divide the proof into two cases, depending on the speed of concentration α_ϵ when compared to ϵ . In section 4.6.1, we treat the case when $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. In section 4.6.2, we will treat the case when $\alpha_\epsilon^2 = O(\epsilon)$.

4.6.1 Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$

We first assume in this subsection that $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. We set $\theta_\epsilon = \frac{\epsilon}{e^{2\hat{v}_l(a)} \alpha_\epsilon^2}$, where $a_\epsilon \rightarrow a \in \mathbb{R}^2$ as $\epsilon \rightarrow 0$, and $i_0 \in \{1, \dots, N_0\}$ such that $\tilde{z}_{i_0} = a$. Then

$$\theta_\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (4.45)$$

We will adapt the technics of Section 4.4.1 to the surface $(\mathbb{S}^2, e^{2\hat{u}_\epsilon} dv_h)$. First, notice that

$$e^{2\hat{u}_\epsilon} dx - d\hat{v}_\epsilon \rightharpoonup_\star 0 \text{ in } \mathcal{M}(\mathbb{R}^2) \text{ as } \epsilon \rightarrow 0. \quad (4.46)$$

Indeed, for $\zeta \in \mathcal{C}_c^0(\mathbb{D}_{R_0})$ for some $R_0 > 0$, and $R > R_0$, we can write that

$$\begin{aligned} \int_{\mathbb{R}^2} \zeta(x) \left(e^{2\hat{u}_\epsilon(x)} dx - d\hat{v}_\epsilon(x) \right) &= \int_{M \setminus \mathbb{D}_R} \left(\int_{\mathbb{D}_{R_0}} p_\epsilon(y, x) \zeta(\hat{y}) dv_g(y) \right) dv_\epsilon(x) \\ &\quad + \int_{\mathbb{D}_R} \left(\int_{\mathbb{D}_R} (\zeta(z) - \zeta(x)) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ &\quad + \int_{\mathbb{D}_{R_0}} \left(\int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz - 1 \right) \zeta(x) d\hat{v}_\epsilon(x). \end{aligned}$$

By estimates (4.9) on the heat kernel, we have that

$$\begin{aligned} \int_{M \setminus \mathbb{D}_R} \left(\int_{\mathbb{D}_{R_0}} p_\epsilon(x, y) |\zeta(\hat{y})| dv_g(y) \right) dv_\epsilon(x) &\leq \|\zeta\|_\infty \sup_{x \in M \setminus \mathbb{D}_R} \int_{\mathbb{D}_{R_0}} p_\epsilon(x, y) dv_g(y) \\ &\leq O \left(\frac{e^{-\frac{(R-R_0)^2}{8\theta_\epsilon}}}{\theta_\epsilon} \right) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

By estimates (4.7) on the heat kernel, we have that

$$\begin{aligned} \int_{\mathbb{D}_R} \left(\int_{\mathbb{D}_R} |\zeta(z) - \zeta(x)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) &\leq \sup_{x \in \mathbb{D}_R} \int_{\mathbb{R}^2} |\zeta(x) - \zeta(z)| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{2\pi\theta_\epsilon} dz \\ &\rightarrow 0 \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

since ζ is uniformly continuous on \mathbb{R}^2 . Finally, we have by the heat kernel estimate (4.10) that

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \sup_{x \in \mathbb{D}_{R_0}} \left| \int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0,$$

so that we get (4.46). We denote by \hat{v} the weak star limit of both $\{e^{2\hat{u}_\epsilon} dx\}$ and $\{\hat{v}_\epsilon\}$ in $\mathcal{M}(\mathbb{R}^2)$.

Let's tackle a generalization of Claim 20 at all the scales which appear between α_ϵ and δ_0 . For a sequence $\{\gamma_\epsilon\}$, we let

$$e^{2\overline{u}_\epsilon^{\gamma_\epsilon}(x)} = \gamma_\epsilon^2 e^{2\hat{u}_\epsilon^l(\gamma_\epsilon x + a_\epsilon)} \text{ and } \overline{\Phi}_\epsilon^{\gamma_\epsilon}(x) = \widetilde{\Phi}_\epsilon^l(\gamma_\epsilon x + a_\epsilon),$$

and for a sequence of domains $\Omega_\epsilon \subset \omega_l$,

$$\overline{\Omega}_\epsilon^{\gamma_\epsilon} = H_{a_\epsilon, \gamma_\epsilon}^{-1} \left(\widetilde{\Omega}_\epsilon^l \right)$$

so that

$$\lambda_\star(\Omega_\epsilon, e^{2u_\epsilon} g) = \lambda_\star(\overline{\Omega}_\epsilon^{\gamma_\epsilon}, e^{2\overline{u}_\epsilon^{\gamma_\epsilon}} \zeta)$$

and

$$\Delta_{\tilde{\zeta}} \overline{\Phi_\epsilon}^{\gamma_\epsilon} = \lambda_\epsilon e^{2\overline{u_\epsilon}^{\gamma_\epsilon}} \overline{\Phi_\epsilon}^{\gamma_\epsilon}.$$

We also let \mathbb{A}_ρ be the annulus $\mathbb{D}_{\frac{1}{\rho}} \setminus \mathbb{D}_\rho$.

We recall that $X_r(\Omega, \{e^{2\overline{u_\epsilon}^{\gamma_\epsilon}} \tilde{\zeta}\})$ is the set of points of $\Omega \subset \mathbb{R}^2$ such that there exists $\epsilon > 0$ which satisfies $\mathbf{P}_{r,\epsilon}$, that is $\mathbf{A}_{r,\epsilon}$ or $\mathbf{B}_{r,\epsilon}$.

$$\mathbf{A}_{r,\epsilon} : \lambda_\star(\mathbb{D}_r(x), e^{2\overline{u_\epsilon}^l}) \leq \frac{\Lambda_k(M, [g])}{2}$$

$\mathbf{B}_{r,\epsilon}$: There exists $f \in E_k(M, e^{2u_\epsilon} g)$ such that $\bar{f}^{\gamma_\epsilon}(x) = 0$ and the Nodal set of $\bar{f}^{\gamma_\epsilon}$ which contains x does not intersect $\partial \mathbb{D}_r(x)$.

Note that for $\gamma_\epsilon = \alpha_\epsilon$, $e^{2\overline{u_\epsilon}^{\gamma_\epsilon}} = e^{2\hat{u}_\epsilon}$ and that the set of concentration points satisfies

$$Z(\Omega, \{e^{2\hat{u}_\epsilon} dx\}) \subset X_r(\Omega, \{e^{2\hat{u}_\epsilon} \tilde{\zeta}\}) \quad (4.47)$$

for all $r > 0$. We write $\omega_1^\epsilon \ll \omega_2^\epsilon$ if two sequences $\{\omega_1^\epsilon\}$ and $\{\omega_2^\epsilon\}$ satisfy $\frac{\omega_1^\epsilon}{\omega_2^\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Claim 26. *Up to the extraction of a subsequence, there exist some sequences $\{\omega_i^\epsilon\}$ with $0 \leq i \leq t+1$ and $0 \leq t \leq k$ such that*

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \cdots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0,$$

there exist $R_0 > 0$ and some points $p_{i,j}$ with $0 \leq i \leq t$ and $1 \leq j \leq s_i$ such that if $1 \leq i \leq t$, $p_{i,j} \in \mathbb{A}_{\frac{1}{R_0}}$ and if $i = 0$, $p_{0,j} \in \mathbb{D}_{R_0}$, with

$$s - 1 + \sum_{i=0}^t s_i \leq k$$

and for all $0 < \rho < \frac{1}{2R_0}$, there exists some $r > 0$ such that for all $1 \leq i \leq t$,

$$X_r \left(\mathbb{A}_\rho, \{e^{2\overline{u_\epsilon}^{\omega_i^\epsilon}} dx\} \right) \subset \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j}),$$

$$X_r \left(\mathbb{D}_{\frac{1}{\rho}}, \{e^{2\hat{u}_\epsilon} dx\} \right) \subset \bigcup_{j=1}^{s_0} \mathbb{D}_\rho(p_{0,j}),$$

for all sequence $\{\gamma_\epsilon\}$ such that $\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho \omega_{i+1}^\epsilon$ with $0 \leq i \leq t$ fixed,

$$X_r \left(\mathbb{A}_{R_0 \rho}, \{e^{2\overline{u_\epsilon}^{\gamma_\epsilon}} dx\} \right) = \emptyset,$$

and for all $0 < \rho < \frac{1}{2R_0}$, for all $r > 0$, for all $0 \leq i \leq t$, $1 \leq j \leq s_i$ and for all subsequence $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$,

$$X_r \left(\mathbb{D}_{\frac{1}{\rho}}, \{e^{2\overline{u_{\epsilon_m}}^{\omega_i^{\epsilon_m}}} dx\}_{m \geq 0} \right) \cap \mathbb{D}_\rho(p_{i,j}) \neq \emptyset. \quad (4.48)$$

Proof. By contradiction, we assume that for all subsequence $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$, for all $\{\omega_i^{\epsilon_m}\}_{m \geq 0}$ with $0 \leq i \leq t$ and

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \cdots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0,$$

for all families of points $p_{i,j} \in \mathbb{R}^2$ with $0 \leq i \leq t$ and $1 \leq j \leq s_i$ such that if $1 \leq i \leq t$, $p_{i,j} \in \mathbb{A}_{\frac{1}{R_0}}$ and if $i = 0$, $p_{0,j} \in \mathbb{D}_{R_0}$, with

$$s - 1 + \sum_{i=0}^t s_i \leq k$$

and

$$R_0 = \max \left\{ \max_{1 \leq i \leq t, 1 \leq j \leq s_i} \left\{ \max \left\{ |p_{i,j}|, \frac{1}{|p_{i,j}|} \right\} \right\}, \max_{1 \leq j \leq s_0} \{|p_{0,j}|\}, \delta_0 \right\} + 1,$$

there exists $0 < \rho < \frac{1}{2R_0}$ such that for all $r > 0$, either there exists $1 \leq i \leq t$ such that

$$X_r \left(\mathbb{A}_\rho, \{e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}} dx\} \right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j}) \neq \emptyset, \quad (4.49)$$

or

$$X_r \left(\mathbb{D}_{\frac{1}{\rho}}, \{e^{2\hat{u}_\epsilon} dx\} \right) \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_\rho(p_{0,j}) \neq \emptyset, \quad (4.50)$$

or there exists a sequence $\{\gamma_\epsilon\}$ such that $\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho \omega_{i+1}^\epsilon$ for some $0 \leq i \leq t$,

$$X_r \left(\mathbb{A}_{R_0 \rho}, \{e^{2\overline{u}_\epsilon^{\gamma_\epsilon}} dx\} \right) \neq \emptyset. \quad (4.51)$$

With this assumption, we prove by induction the following property $\mathbf{H}_{\tilde{s}}$ for $s - 1 \leq \tilde{s} \leq k + 1$

$\mathbf{H}_{\tilde{s}}$: there exists sequences $\epsilon_m \rightarrow 0$ and $r_m \searrow 0$ as $m \rightarrow +\infty$, some scales

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \cdots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0,$$

some points $p_{i,j}^m, p_{i,j} \in \mathbb{R}^2 \setminus \{0\}$ if $1 \leq i \leq t$, $1 \leq j \leq s_i$ and $p_{0,j}^m, p_{0,j} \in \mathbb{R}^2$ if $1 \leq j \leq s_0$ with

$$s - 1 + \sum_{i=0}^t s_i = \tilde{s}$$

and $p_{i,j} \neq p_{i,j'}$ if $j \neq j'$ for $0 \leq i \leq t$, such that for all $0 \leq i \leq t$, $1 \leq j \leq s_i$, $p_{i,j}^m$ satisfies $\mathbf{P}_{r_m, \epsilon_m}$ in $(\mathbb{R}^2, \{e^{2\overline{u}_{\epsilon_m}^{\omega_i^{\epsilon_m}}} \xi\}_{m \geq 0})$.

We already have \mathbf{H}_{s-1} , let's prove \mathbf{H}_s . We fix $\rho > 0$. By assumption, since we apply it with all s_i 's equal to 0, either (4.51) or (4.50) happen. Let's study these two cases :

CASE (4.51) _{$s-1$} : There exists a sequence $\{\gamma_{\epsilon_m}\}$ with $\frac{\alpha_{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho \delta_0$ and some $x_m \in X_{2^{-m}} \left(\mathbb{A}_\rho, \{e^{2\overline{u}_\epsilon^{\gamma_\epsilon}} dx\} \right)$. We choose ϵ_m such that x_m satisfies $\mathbf{P}_{2^{-m}, \epsilon_m}$. It is clear that $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$.

- If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}} \rightarrow +\infty$, we set a new scale $\omega_1^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{1,1}^m = x_m \in \mathbb{A}_\rho$. Up to the extraction of a subsequence, $p_{1,1}^m \rightarrow p_{1,1} \in \mathbb{R}^2 \setminus \{0\}$ as $m \rightarrow +\infty$. It is clear by Claim 20 that $\omega_1^{\epsilon_m} \ll \delta_0$ up to reduce ρ , and we get \mathbf{H}_s in this case.
- If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}}$ is bounded, up to reduce ρ , one gets that (4.50) holds and we can go to Case (4.50)_{s-1}.

CASE (4.50)_{s-1} : There exists $x_m \in X_{2^{-m}} \left(\mathbb{D}_{\frac{1}{\rho}}, \{e^{2u_{\epsilon}} dx\} \right)$. We set $p_{0,1}^m = x_m$ and up to the extraction of a subsequence, $p_{0,1}^m \rightarrow p_{0,1}$ as $m \rightarrow +\infty$ and we get \mathbf{H}_s in this case.

Now, we assume that $\mathbf{H}_{\tilde{s}}$ is true for some $s \leq \tilde{s} \leq k$. Let's prove $\mathbf{H}_{\tilde{s}+1}$. We define all the parameters $\epsilon_m, r_m, \omega_i^{\epsilon_m}, p_{i,j}^m$ and $p_{i,j}$ given by $\mathbf{H}_{\tilde{s}}$. We fix $\rho > 0$. By assumption, one of the assertions (4.49), (4.50) and (4.51) must happen. Let's study these three cases :

CASE (4.49) _{\tilde{s}} : Let $1 \leq i \leq t$ and $x_m \in X_{r_m} \left(\mathbb{A}_\rho, \{e^{2u_{\epsilon_m} \omega_i^{\epsilon_m}} dx\} \right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j})$. For $m \geq 0$, we set $p_{i,s_i+1}^m = x_m$ and we let $\epsilon_{\beta(m)}$ be such that p_{i,s_i+1}^m satisfies $\mathbf{P}_{r_m, \epsilon_{\beta(m)}}$. Since $r_m \searrow 0$, as $m \rightarrow +\infty$, setting $M(m) = \min\{m, \beta(m)\}$ gives that $p_{i,j}^{\beta(m)}$ satisfies $\mathbf{P}_{r_{M(m)}, \epsilon_{\beta(m)}}$ for all $1 \leq j \leq s_i$ and $p_{i,s_i+1}^{\beta(m)}$ satisfies $\mathbf{P}_{r_{M(m)}, \epsilon_{\beta(m)}}$. Up to the extraction of a subsequence, we can assume that $p_{i,s_i+1}^m \rightarrow p_{i,s_i+1}$ as $m \rightarrow +\infty$ and that $r_{M(m)} \searrow 0$ as $m \rightarrow +\infty$. Since $p_{i,s_i+1}^m \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j})$, $p_{i,s_i+1} \in \mathbb{R}^2 \setminus \{0, p_{i,1}, \dots, p_{i,s_i}\}$. The proof of $\mathbf{H}_{\tilde{s}+1}$ is complete in this case.

CASE (4.50) _{\tilde{s}} : The proof of $\mathbf{H}_{\tilde{s}+1}$ is the same as in (4.49) _{\tilde{s}} .

CASE (4.51) _{\tilde{s}} : Let $\{\gamma_{\epsilon_m}\}$ be a sequence such that

$$\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho \omega_{i+1}^{\epsilon_m} \text{ and } x_m \in X_{r_m} \left(\mathbb{A}_{R_0 \rho}, \{e^{2u_{\epsilon_q} \gamma_{\epsilon_q}} dx\}_{q \geq 0} \right).$$

- If $\frac{\gamma_{\epsilon_m}}{\omega_i^{\epsilon_m}} \rightarrow +\infty$ and $\frac{\gamma_{\epsilon_m}}{\omega_{i+1}^{\epsilon_m}} \rightarrow 0$, we define a new scale $\omega_{t+1}^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{t+1,1}^m = x_m$. Up to the extraction of a subsequence, $p_{t+1,1}^m \in \mathbb{A}_\rho$ satisfies $\mathbf{P}_{r_m, \epsilon_m}$, $p_{t+1,1}^m \rightarrow p_{t+1,1} \in \mathbb{R}^2 \setminus \{0\}$ and $r_m \searrow 0$ as $m \rightarrow +\infty$. Up to reorder $\{\omega_i^{\epsilon_m}\}$, we get $\mathbf{H}_{\tilde{s}+1}$ in this case.
- If $i = 0$ and $\frac{\gamma_{\epsilon_m}}{\omega_0^{\epsilon_m}}$ is bounded, up to reduce ρ , we get that (4.50) holds and go back to Case (4.50) _{\tilde{s}} .
- The case $i = t$ and $\frac{\omega_{t+1}^{\epsilon_m}}{\gamma_{\epsilon_m}}$ is bounded leads to a contradiction by Claim 20.
- The other cases lead to the fact that (4.49) holds up to reduce ρ and we are back to Case (4.49) _{\tilde{s}} .

Gathering the three cases, we deduce $\mathbf{H}_{\tilde{s}+1}$. Therefore, \mathbf{H}_{k+1} holds true and we now prove that this leads to a contradiction. We will define new test functions for the variational characterization (4.2.1) of $\lambda_k(e^{2u_\epsilon} g)$ on $M, \eta_{i,j}^m$ for $0 \leq i \leq t, 1 \leq j \leq s_i$.

- If $p_{i,j}^m$ satisfies $\mathbf{A}_{r_m, \epsilon_m}$, $\eta_{i,j}^m$ is defined by an eigenfunction for $\lambda_\star \left(\Omega_{i,j}^m, e^{2u_{\epsilon_m}} g \right)$ extended by 0 in $M \setminus \Omega_{i,j}^m$, where $\Omega_{i,j}^m \subset M$ is defined by $\mathbb{D}_{r_m}(p_{i,j}^m) = \overline{\Omega_{i,j}^m}^{\omega_i^{\epsilon_m}}$.
- If $p_{i,j}^m$ does not satisfy $\mathbf{A}_{r_m, \epsilon_m}$, it satisfies $\mathbf{B}_{r_m, \epsilon_m}$ and $\eta_{i,j}^m$ is defined by an eigenfunction for $\lambda_\star \left(D_{i,j}^m, e^{2u_{\epsilon_m}} g \right)$ extended by 0 in $M \setminus D_{i,j}^m$, where $D_{i,j}^m \subset M$ is the Nodal domain of an eigenfunction associated to λ_{ϵ_m} such that $\overline{D_{i,j}^m}^{\omega_i^{\epsilon_m}} \subset \mathbb{D}_{r_m}(p_{i,j}^m)$.

We also use the functions η_i^m for $\{1 \leq i \leq s\}$, already defined in the proof of Claim 20. Note that these $k + 1$ functions have pairwise disjoint support for m large enough. Then

$$\lambda_{\epsilon_m} \leq \max \left\{ \max_{0 \leq i \leq t, 1 \leq j \leq s_i} \frac{\int_M |\nabla \eta_{i,j}^m|_g^2 dv_g}{\int_M (\eta_{i,j}^m)^2 e^{2u_{\epsilon_m}} dv_g}, \max_{i \neq i_0} \frac{\int_M |\nabla \eta_i^m|_g^2 dv_g}{\int_M (\eta_i^m)^2 e^{2u_{\epsilon_m}} dv_g} \right\} \leq \lambda_{\epsilon_m}.$$

The last inequality comes from the definition of the properties **A** and **B** and we have equality if and only if one of the test functions is an eigenfunction for $\lambda_{\epsilon_m} = \lambda_k(M, e^{2u_{\epsilon_m}} g)$. This contradicts the maximum principle since the test functions vanish on open sets of the manifold.

Therefore we proved the first part of the claim. Up to make successive extractions of subsequences of $\{\epsilon_m\}$ and up to remove some points $p_{i,j}$, one easily proves that the last condition (4.48) also holds. ◇

For $\rho > 0$, we set

$$\Omega(\rho) = \mathbb{D}_{\frac{1}{\rho}} \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_{\rho}(p_{0,j})$$

As previously remarked, the set of concentration points of $\{e^{2\hat{u}_{\epsilon}} dx\}$ satisfies

$$Z(\mathbb{R}^2, \{e^{2\hat{u}_{\epsilon}} dx\}) \subset \{p_{0,1}, \dots, p_{0,s_0}\} \quad (4.52)$$

and letting

$$m_i(\rho) = \lim_{\epsilon \rightarrow 0} \int_{\Omega(\rho)} e^{2\hat{u}_{\epsilon}},$$

we have that $m_i(\rho) \geq m_i + o(1) > 0$ since we have (4.35), (4.37) (4.52) and $m_i > 0$. We aim at getting regularity estimates on $\hat{\Phi}_{\epsilon}$ and $e^{2\hat{u}_{\epsilon}}$ in $\Omega(\rho)$. We follow the proof of Claim 21, thanks to the fact that $m_i(\rho) > 0$ for ρ small enough.

Claim 27. *We have the following*

— *Estimates on $\hat{\Phi}_{\epsilon}$:*

$$\forall \rho > 0, \exists C_1(\rho) > 0, \forall \epsilon > 0, \|\hat{\Phi}_{\epsilon}\|_{W^{1,2}(\Omega(\rho))} \leq C_1(\rho), \quad (4.53)$$

$$\forall \rho > 0, \exists C_2(\rho) > 0, \forall \epsilon > 0, \|\hat{\Phi}_{\epsilon}\|_{C^0(\Omega(\rho))} \leq C_2(\rho). \quad (4.54)$$

— *Quantitative non-concentration estimates on $e^{2\hat{u}_{\epsilon}}$ and $|\nabla \hat{\Phi}_{\epsilon}|^2$:*

$$\forall \rho > 0, \exists D_1(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_r(x)} e^{2\hat{u}_{\epsilon}} \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})}, \quad (4.55)$$

$$\forall \rho > 0, \exists D_2(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_r(x)} |\nabla \hat{\Phi}_{\epsilon}|^2 \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}}. \quad (4.56)$$

Proof.

We first prove (4.53) using Claim 26 and the fact that $m_i(\rho) > 0$. Let's prove that $\{\frac{e^{2\hat{u}_{\epsilon}}}{\int_{\Omega(\rho)} e^{2\hat{u}_{\epsilon}}}\}$ is a bounded sequence in $W^{-1,2}(\Omega(\rho))$. Let $\rho > 0$ and $r > 0$ such that $X_r(\Omega(\rho), \{e^{2\hat{u}_{\epsilon}} dx\}) = \emptyset$. Then, for all $x \in \Omega(\rho)$, and $\epsilon > 0$, $\lambda_{\star}(\mathbb{D}_r(x), e^{2\hat{u}_{\epsilon}} \xi) > \frac{\Lambda_k(M, [g])}{2}$.

By compactness, we let $y_1, \dots, y_K \in \Omega(\rho)$ be such that $\Omega(\rho) \subset \bigcup_{i=1}^K \mathbb{D}_r(y_i)$ and ψ_1, \dots, ψ_K a partition of unity associated to this covering so that $\sum_{i=1}^K \psi_i = 1$ on $\Omega(\rho)$ and $\text{supp}(\psi_i) \subset \mathbb{D}_r(y_i)$. Let $L : W^{1,2}(\Omega(\rho)) \rightarrow W^{1,2}(\mathbb{R})$ be a continuous extension operator. Then, if $\psi \in W^{1,2}(\Omega(\rho))$,

$$\begin{aligned}
 \int_{\Omega(\rho)} \psi \frac{e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} &= \sum_{i=1}^K \int_{\Omega(\rho) \cap \mathbb{D}_r(y_i)} \psi \psi_i \frac{e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \\
 &\leq \sum_{i=1}^K \left(\int_{\Omega(\rho) \cap \mathbb{D}_r(y_i)} (\psi \psi_i)^2 \frac{e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^K \left(\int_{\mathbb{D}_r(y_i)} (L(\psi) \psi_i)^2 \frac{e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^K \frac{\left(\int_{\mathbb{D}_r(y_i)} |\nabla (L(\psi) \psi_i)|^2 dx \right)^{\frac{1}{2}}}{\lambda_\star(\mathbb{D}_r(y_i), e^{\hat{u}_\epsilon} \zeta)^{\frac{1}{2}} \left(\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx \right)^{\frac{1}{2}}} \\
 &\leq \frac{A_0(\rho)}{\left(\frac{\Lambda_k(M, [g])}{2} \right)^{\frac{1}{2}} m_i(\rho)^{\frac{1}{2}}} \|L(\psi)\|_{W^{1,2}} \\
 &\leq A_1(\rho) \|\psi\|_{W^{1,2}(\Omega(\rho))}
 \end{aligned}$$

for some constants $A_0(\rho)$ and $A_1(\rho)$ which do not depend on $\epsilon > 0$. By the Poincaré inequality of Theorem 10, there exists some constant $A_2(\rho) > 0$ such that for $f \in \mathcal{C}^\infty(\Omega(\rho))$

$$\forall \epsilon > 0, \int_{\Omega(\rho)} \left(f - \int_{\Omega(\rho)} f \frac{e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \right)^2 dx \leq A_2(\rho) \int_{\Omega(\rho)} |\nabla f|^2 dx.$$

We deduce from this inequality that

$$\|f\|_{L^2(\Omega(\rho))} \leq \left(A_2(\rho) \int_{\Omega(\rho)} |\nabla f|^2 dx + \frac{\int_{\Omega(\rho)} f^2 e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \right)^{\frac{1}{2}} + \left(\frac{\int_{\Omega(\rho)} f^2 e^{2\hat{u}_\epsilon} dx}{\int_{\Omega(\rho)} e^{2\hat{u}_\epsilon} dx} \right)^{\frac{1}{2}}.$$

Since $m_i(\rho) > 0$, applying this inequality to $\hat{\phi}_\epsilon^i$ gives (4.53).

We now prove (4.54). Let $1 \leq i \leq n+1$. Up to change $\hat{\phi}_\epsilon^i$ into $-\hat{\phi}_\epsilon^i$, there exists a subsequence $\{x_\epsilon\}$ of points in $\Omega(\rho)$ such that $\hat{\phi}_\epsilon^i(x_\epsilon) = \sup_{\Omega(\rho)} |\hat{\phi}_\epsilon^i|$. We set $\delta_\epsilon = d_\zeta(x_\epsilon, \text{supp}(\hat{v}_\epsilon))$ and we let $y_\epsilon \in \text{supp}(\hat{v}_\epsilon)$ be such that $\delta_\epsilon = |x_\epsilon - y_\epsilon|$. We divide the proof into 3 cases :

CASE 1 - $\delta_\epsilon^{-1} = O(1)$. Then, $\{e^{2\hat{u}_\epsilon}\}$ is uniformly bounded in $\mathbb{D}_{\min(\frac{\delta_\epsilon}{2}, \frac{\rho}{2})}(x_\epsilon)$ by estimates on the heat kernel (see (4.8)). By (4.53), $\hat{\phi}_\epsilon^i$ is bounded in $L^2(\Omega(\frac{\rho}{2}))$ and $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded by standard elliptic theory on the equation (4.44).

CASE 2 - $\delta_\epsilon = O\left(\frac{\sqrt{\epsilon}}{\alpha_\epsilon}\right)$. Using Claim 19, we get that $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded.

CASE 3 - $\delta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\frac{\sqrt{\epsilon}}{\alpha_\epsilon \delta_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. We set

$$e^{2w_\epsilon(x)} = \delta_\epsilon^2 e^{2\hat{u}_\epsilon(x_\epsilon + \delta_\epsilon x)}$$

$$\begin{aligned}\psi_\epsilon(x) &= \phi_\epsilon^i(x_\epsilon + \delta_\epsilon x) \\ z_\epsilon &= \frac{1}{\delta_\epsilon}(x_\epsilon - y_\epsilon)\end{aligned}$$

so that

$$\Delta\psi_\epsilon = \lambda_\epsilon e^{2w_\epsilon} \psi_\epsilon \text{ in } \mathbb{D}_5. \quad (4.57)$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R}^2$ with $|z_0| = 1$ such that $z_\epsilon \rightarrow z_0$ as $\epsilon \rightarrow 0$. By estimates (4.8), there is $D_1 > 0$ such that

$$e^{2w_\epsilon} \leq D_1 \text{ in } \mathbb{D}_{\frac{1}{2}}.$$

By Claim 19, since $y_\epsilon \in \text{supp}(\hat{v}_\epsilon)$, $\psi_\epsilon(z_\epsilon) = O(1)$ as $\epsilon \rightarrow 0$.

We first assume that ψ_ϵ does not vanish in $\mathbb{D}_3(0)$. Since $\psi_\epsilon(0) > 0$, $\psi_\epsilon > 0$ in $\mathbb{D}_3(0)$. Then, by Harnack's inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}, \psi_\epsilon(x) \geq D_2 \psi_\epsilon(0).$$

Since ψ_ϵ is positive, ψ_ϵ is superharmonic in $\mathbb{D}_{|z_\epsilon|}(z_\epsilon) \subset \mathbb{D}_3(0)$ by the equation (4.57) so that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{2\pi |z_\epsilon|} \int_{\partial \mathbb{D}_{|z_\epsilon|}(z_\epsilon)} \psi_\epsilon d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that $\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0)$. We conclude that $\phi_\epsilon^i(x_\epsilon) = \psi_\epsilon(0)$ is bounded.

We now assume that ψ_ϵ vanishes in $\mathbb{D}_3(0)$. Since $X_r(\Omega(\rho), e^{2\hat{u}_\epsilon}) = \emptyset$ by Claim 26, ψ_ϵ vanishes in $\mathbb{D}_4(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 11 of Section 4.2.3 for $\Omega = \mathbb{D}_5(0)$, we get some constant $C_1 > 0$ such that

$$\int_{\mathbb{D}_4(0)} \psi_\epsilon^2 dx \leq C_1 \int_{\mathbb{D}_5(0)} |\nabla \psi_\epsilon|^2 dx.$$

By elliptic estimates on (4.57), $\{\psi_\epsilon\}$ is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}(0)$ and $\phi_\epsilon^i(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now tackle the estimate (4.55). Let $r_1 > 0$ be such that for all $0 < r < r_1(\rho)$,

$$X_r(\Omega(\rho), \{e^{2\hat{u}_\epsilon} \tilde{\zeta}\}) = \emptyset.$$

Then,

$$\forall \epsilon > 0, \forall x \in \Omega(\rho), \frac{1}{\lambda_\star(\mathbb{D}_r(x))} \leq \frac{2}{\Lambda_k(M, [g])}.$$

By isocapacity estimates (see [77], section 2.3.3, Theorem 2.3.3),

$$\int_{\mathbb{D}_r(x)} e^{2\hat{u}_\epsilon} dx \leq \frac{\text{cap}_2(\mathbb{D}_r(x), \mathbb{D}_{r_1}(x))}{\lambda_\star(\mathbb{D}_r(x), e^{2\hat{u}_\epsilon} \tilde{\zeta})} \leq \frac{4\pi}{\Lambda_k(M, [g]) \ln\left(\frac{r_1}{r}\right)}.$$

And we get (4.55). The last estimate (4.56) is a consequence of (4.55) as proved in [91], Claim 7.

◇

We now need an estimate of $\{\Phi_\epsilon\}$ on the whole surface in order to prove later that no energy is lost in the necks.

Claim 28. For any $\rho > 0$, there exists a constant $C_0(\rho) > 0$ such that

$$\forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i, \rho) \cup \bigcup_{i=0}^t \bigcup_{j=1}^{s_i} \Omega_{i,j} \right),$$

$$|\Phi_\epsilon|(x) \leq C_0(\rho) \left(\ln \left(1 + \frac{d_g(x, \bar{a}_\epsilon)}{\alpha_\epsilon} \right) + 1 \right),$$

where

$$\tilde{\Omega}_{i,j}^l = \omega_i^\epsilon \mathbb{D}_\rho(p_{i,j}) + a_\epsilon \text{ and } \bar{a}_\epsilon = \exp_{g_l, x_l}^{-1}(a_\epsilon).$$

Proof. Let $0 < \rho < \frac{1}{20R_0}$ and let $r > 0$ which satisfies the conclusion of Claim 26 for this ρ .

STEP 1 : We have that for $0 \leq i \leq t$, there exists $A_i(\rho) > 0$ such that for all $1 \leq \beta \leq n+1$, for all sequence $\{\gamma_\epsilon\}$ with $\frac{\omega_i^\epsilon}{\rho} \leq \gamma_\epsilon \leq \rho\omega_{i+1}^\epsilon$, either

$$\forall x \in \mathbb{A}_{10R_0\rho}, \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right| \leq A_i(\rho)$$

or

$$\forall x, y \in \mathbb{A}_{10R_0\rho}, \frac{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|}{A_i(\rho)} \leq \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right| \leq A_i(\rho) \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|.$$

We let $A_i(\rho)$ be such that

$$\max_{1 \leq \beta \leq n+1} \sup_{\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho\omega_{i+1}^\epsilon} \sup_{\epsilon > 0} \left\{ \min \left\{ \max_{x \in \mathbb{A}_{10R_0\rho}} \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right|, \max_{x, y \in \mathbb{A}_{10R_0\rho}} \frac{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right|}{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|} \right\} \right\}$$

and we assume by contradiction that $A_i(\rho) = +\infty$. Then there exist $1 \leq \beta \leq n+1$, $\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho\omega_{i+1}^{\epsilon_m}$ such that $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$ and

$$\min \left\{ \max_{x \in \mathbb{A}_{10R_0\rho}} \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|, \max_{x, y \in \mathbb{A}_{10R_0\rho}} \frac{\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|}{\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(y) \right|} \right\} \rightarrow +\infty \text{ as } m \rightarrow +\infty.$$

Let $x_m \in \mathbb{A}_{10R_0\rho}$ be such that $\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) \right| = \max_{x \in \mathbb{A}_{10R_0\rho}} \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|$. We set

$$\delta_m = d(x_m, \text{supp}(\overline{v_{\epsilon_m}^{\gamma_{\epsilon_m}}}))$$

and take $y_m \in \text{supp}(\overline{v_{\epsilon_m}^{\gamma_{\epsilon_m}}})$ such that $|x_m - y_m| = \delta_m$. We study 3 cases, each one leading to a contradiction.

CASE 1 - $\delta_m = O\left(\frac{\sqrt{\epsilon_m}}{\gamma_{\epsilon_m}}\right)$. We apply Claim 19 for the sequence of points $\{\exp_{g_l, x_l}(\gamma_{\epsilon_m} x_m + a_{\epsilon_m})\}_m$ in M and we get a contradiction.

CASE 2 - $\delta_m \rightarrow 0$ and $\frac{\delta_m \gamma_{\epsilon_m}}{\sqrt{\epsilon_m}} \rightarrow +\infty$ as $m \rightarrow +\infty$. We set

$$\begin{aligned} e^{2w_m} &= \delta_m^2 e^{\overline{2u_{\epsilon_m}} \gamma_{\epsilon_m}} (x_m + \delta_m x), \\ \psi_m &= \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}} (x_m + \delta_m x), \\ z_m &= \frac{1}{\delta_m} (y_m - x_m), \end{aligned}$$

so that

$$\Delta \psi_m = \lambda_{\epsilon_m} e^{2w_m} \psi_m.$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R}^2$ with $|z_0| = 1$ such that $z_m \rightarrow z_0$ as $m \rightarrow +\infty$. By (4.8), there is $D_1 > 0$ such that

$$e^{2w_m} \leq D_1 \text{ on } \mathbb{D}_{\frac{1}{2}}.$$

By Claim 19, since $y_m \in \text{supp}(\overline{v_{\epsilon_m}}^{\gamma_{\epsilon_m}})$, $\psi_m(z_m) = O(1)$ as $m \rightarrow +\infty$.

We first assume that ψ_m does not vanish in $\mathbb{D}_3(0)$. Up to take $-\psi_m$, we can assume that $\psi_m > 0$ on $\mathbb{D}_3(0)$. Then, by Harnack inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}, \psi_m(x) \geq D_2 \psi_m(0).$$

Since ψ_m is positive, ψ_m is superharmonic in $\mathbb{D}_{|z_m|}(z_m) \subset \mathbb{D}_3(0)$. Then,

$$\psi_m(z_m) \geq \frac{1}{2\pi |z_m|} \int_{\partial \mathbb{D}_{|z_m|}(z_m)} \psi_m d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that

$\psi_m(z_m) \geq D_3 \psi_m(0)$. We conclude that $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

We assume now that ψ_m vanishes in $\mathbb{D}_3(0)$. By Claim 26, ψ_m vanishes in $\mathbb{D}_4(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 11 for $\Omega = \mathbb{D}_5(0)$, we get a Poincaré inequality

$$\int_{\mathbb{D}_4(0)} \psi_m^2 dx \leq C_1 \int_{\mathbb{D}_5(0)} |\nabla \psi_m|^2 dx.$$

By elliptic regularity theory, ψ_m is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}(0)$ and $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

CASE 3 - $\frac{1}{\delta_m} = O(1)$. Up to the extraction of a subsequence, we assume that $x_m \rightarrow x$ in $\mathbb{A}_{10R_0\rho}$ as $m \rightarrow +\infty$.

We first assume that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}$ vanishes in $\mathbb{A}_{5R_0\rho}$. We get by Claim 26 and the corollary of Theorem 11 for $\Omega = \mathbb{A}_{2R_0\rho}$ a constant $C_r > 0$ such that

$$\int_{\mathbb{A}_{4R_0\rho}} \psi_m^2 dx \leq C_r \int_{\mathbb{A}_{2R_0\rho}} |\nabla \psi_m|^2 dx.$$

By (4.8), there are some constants $\tilde{r} > 0$ and $D_1 > 0$ such that

$$e^{\overline{2u_{\epsilon_m}} \gamma_{\epsilon_m}} \leq D_1 \text{ on } \mathbb{D}_{\tilde{r}}(x).$$

By elliptic estimates, $\{\psi_m\}$ is uniformly bounded on $\mathbb{A}_{5R_0\rho} \cap \mathbb{D}_{\frac{\delta}{2}}(x)$ which gives a contradiction.

We assume now that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}$ does not vanish in $\mathbb{A}_{5R_0\rho}$. Up to take $-\psi_m$, we assume that $\psi_m > 0$ on $\mathbb{A}_{5R_0\rho}$.

Let's assume that $y_m \rightarrow y$ as $m \rightarrow +\infty$ with $y \in \mathbb{A}_{7R_0\rho}$. By Claim 19, $\psi_m(y_m) = O(1)$. By (4.8), there exists a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } \mathbb{D}_{\delta-\tilde{\delta}}(x),$$

where $\tilde{\delta} = \min\left(\frac{\delta}{4}, \frac{R_0\rho}{4}\right)$. By Harnack's inequality, there exists $D_2 > 0$ such that

$$\forall z \in \mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}(x), \psi_m(x_m) \leq D_2 \psi_m(z).$$

By superharmonicity on $\mathbb{D}_{3\tilde{\delta}}(y_m) \subset \mathbb{A}_{5R_0\rho}$,

$$\psi_m(y_m) \geq \frac{1}{2\pi \times 3\tilde{\delta}} \int_{\partial \mathbb{D}_{3\tilde{\delta}}(y_m)} \psi_m d\sigma.$$

We keep the part of the integral which lies in $\mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$. Since the length of $\partial \mathbb{D}_{3\tilde{\delta}}(y_m) \cap \mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$ is uniformly bounded from below, we get a constant $D_3 > 0$ such that $\psi_m(y_m) \geq D_3 \psi_m(x_m)$. Then, $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(x_m) = O(1)$ which is absurd.

Assume now that $y_m \in \mathbb{R}^2 \setminus \mathbb{A}_{8R_0\rho}$. By (4.8), there is a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } \mathbb{A}_{9R_0\rho}.$$

By Harnack inequality, there exists a constant $C_1 > 0$ such that

$$\forall z \in \mathbb{A}_{10R_0\rho}, |\psi_m|(x_m) \leq C_1 |\psi_m|(z).$$

By definition of x_m , we get

$$\forall z, \tilde{z} \in \mathbb{A}_{10R_0\rho}, \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(\tilde{z}) \right| \leq \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) \right| \leq C_1 \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(z) \right|,$$

which also leads to a contradiction.

STEP 2 : We have that for $1 \leq i \leq t$, there exists $B_i(\rho) > 0$ such that for all $1 \leq \beta \leq n+1$, either

$$\forall x \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j}), \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right| \leq B_i(\rho)$$

or

$$\forall x, y \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j}), \frac{\left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(y) \right|}{B_i(\rho)} \leq \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right| \leq B_i(\rho) \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(y) \right|$$

We set

$$B_i(\rho) = \max_{1 \leq \beta \leq n+1} \sup_{\epsilon > 0} \left\{ \min \left\{ \max_{x \in \mathbb{U}_\rho} \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right|, \max_{x, y \in \mathbb{U}_\rho} \frac{\left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right|}{\left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(y) \right|} \right\} \right\}$$

with

$$\mathbb{U}_\rho = \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(p_{i,j}).$$

We prove that $B_i(\rho) < +\infty$. Let $1 \leq \beta \leq n+1$, a subsequence $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$ and let $x_m \in \mathbb{U}_\rho$ be such that $\left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}(x_m) \right| = \max_{x \in \mathbb{U}_\rho} \left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}(x) \right|$. We set $\delta_m = d(x_m, \text{supp}(\overline{v_{\epsilon_m}}^{\omega_i^{\epsilon_m}}))$ and take $y_m \in \text{supp}(\overline{v_{\epsilon_m}}^{\omega_i^{\epsilon_m}})$ such that $|x_m - y_m| = \delta_m$. We consider 3 cases.

CASE 1 - $\delta_m = O\left(\frac{\sqrt{\epsilon_m}}{\omega_i^{\epsilon_m}}\right)$. We apply Claim 19 for the sequence : $\{\exp_{g_l, x_l}(\omega_i^{\epsilon_m} x_m + a_{\epsilon_m})\}_m$ of points in M and we get a uniform bound for $\left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}(x_m) \right|$.

CASE 2 - $\delta_m \rightarrow 0$ and $\frac{\delta_m \gamma_{\epsilon_m}}{\sqrt{\epsilon_m}} \rightarrow +\infty$ as $m \rightarrow +\infty$. We set

$$\begin{aligned} e^{2w_m} &= \delta_m^2 e^{2\overline{u_{\epsilon_m}}^{\omega_i^{\epsilon_m}}}(x_m + \delta_m x), \\ \psi_m &= \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}(x_m + \delta_m x), \\ z_m &= \frac{1}{\delta_m}(y_m - x_m), \end{aligned}$$

so that

$$\Delta \psi_m = \lambda_{\epsilon_m} e^{2w_m} \psi_m.$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R}^2$ with $|z_0| = 1$ such that $z_m \rightarrow z_0$ as $m \rightarrow +\infty$. By (4.8), there is $D_1 > 0$ such that

$$e^{2w_m} \leq D_1 \text{ on } \mathbb{D}_{\frac{1}{2}}.$$

By Claim 19, since $y_m \in \text{supp}(\overline{v_{\epsilon_m}}^{\gamma_{\epsilon_m}})$, $\psi_m(z_m) = O(1)$ as $m \rightarrow +\infty$.

We first assume that ψ_m does not vanish in $\mathbb{D}_3(0)$. Up to take $-\psi_m$, we may assume that $\psi_m > 0$ on $\mathbb{D}_3(0)$. Then, by Harnack inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}, \psi_m(x) \geq D_2 \psi_m(0).$$

Since ψ_m is positive, ψ_m is superharmonic in $\mathbb{D}_{|z_m|}(z_m) \subset \mathbb{D}_3(0)$. Then,

$$\psi_m(z_m) \geq \frac{1}{2\pi |z_m|} \int_{\partial \mathbb{D}_{|z_m|}(z_m)} \psi_m d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that

$$\psi_m(z_m) \geq D_3 \psi_m(0). \text{ We conclude that } \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1).$$

We assume now that ψ_m vanishes in $\mathbb{D}_3(0)$. By Claim 26, ψ_m vanishes in $\mathbb{D}_4(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 11 for $\Omega = \mathbb{D}_5(0)$, we get a Poincaré inequality

$$\int_{\mathbb{D}_4(0)} \psi_m^2 dx \leq C_1 \int_{\mathbb{D}_5(0)} |\nabla \psi_m|^2 dx.$$

By elliptic regularity, ψ_m is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}(0)$ and $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$.

CASE 3 - $\frac{1}{\delta_m} = O(1)$. Up to the extraction of a subsequence, we assume that $x_m \rightarrow x$ in \mathbb{U}_ρ as $m \rightarrow +\infty$.

We first assume that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}$ vanishes in $\mathbb{U}_{\frac{\rho}{2}}$. We get with Claim 26 and the corollary of Theorem 11 for $\Omega = \mathbb{U}_{\frac{\rho}{4}}$ a constant $C_r > 0$ such that

$$\int_{\mathbb{U}_{\frac{\rho}{3}}} \psi_m^2 dx \leq C_r \int_{\mathbb{U}_{\frac{\rho}{4}}} |\nabla \psi_m|^2 dx.$$

By (4.8), there are some constants $\tilde{r} > 0$ and $D_1 > 0$ such that

$$e^{2\overline{u_{\epsilon_m}} \omega_i^{\epsilon_m}} \leq D_1 \text{ on } \mathbb{D}_{\tilde{r}}(x).$$

By elliptic estimates, $\{\psi_m\}$ is uniformly bounded on $\mathbb{U}_{\frac{\rho}{2}} \cap \mathbb{D}_{\frac{\tilde{r}}{2}}(x)$ so that $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = O(1)$.

We assume now that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}$ does not vanish in $\mathbb{U}_{\frac{\rho}{2}}$. Up to take $-\psi_m$, we may assume that $\psi_m > 0$ on $\mathbb{U}_{\frac{\rho}{2}}$.

Let's assume that $y_m \rightarrow y$ as $m \rightarrow +\infty$ with $y \in \mathbb{U}_{\frac{7\rho}{10}}$. By Claim 19, $\psi_m(y_m) = O(1)$. By (4.8), there exists a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\epsilon_m}} \omega_i^{\epsilon_m}} \leq D_1 \text{ in } \mathbb{D}_{\delta-\tilde{\delta}}(x),$$

where $\tilde{\delta} = \min(\frac{\delta}{4}, \frac{\rho}{40})$. By Harnack's inequality, there exists $D_2 > 0$ such that

$$\forall z \in \mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}(x), \psi_m(x_m) \leq D_2 \psi_m(z).$$

By superharmonicity on $\partial \mathbb{D}_{3\tilde{\delta}}(y_m) \subset \mathbb{U}_{\frac{\rho}{2}}$,

$$\psi_m(y_m) \geq \frac{1}{2\pi \times 3\tilde{\delta}} \int_{\partial \mathbb{D}_{3\tilde{\delta}}(y_m)} \psi_m d\sigma$$

We keep the part of the integral which lies in $\mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$ since the length of $\partial \mathbb{D}_{3\tilde{\delta}}(y_m) \cap \mathbb{U}_{\frac{6\rho}{10}} \cap \mathbb{D}_{\delta-2\tilde{\delta}}$ is uniformly bounded below and we get a constant $D_3 > 0$ such that $\psi_m(y_m) \geq$

$D_3 \psi_m(x_m)$. Then, $\overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}}(x_m) = \psi_m(x_m) = O(1)$.

Assume now that $y_m \in \mathbb{R}^2 \setminus \mathbb{U}_{\frac{8\rho}{10}}$. By (4.8), there is a constant $D_1 > 0$ such that

$$e^{2\overline{u_{\epsilon_m}} \omega_i^{\epsilon_m}} \leq D_1 \text{ in } \mathbb{U}_{\frac{9\rho}{10}}.$$

By Harnack inequality, there exists a constant $C_1 > 0$ such that

$$\forall z \in \mathbb{U}_\rho, |\psi_m|(x_m) \leq C_1 |\psi_m|(z)$$

By definition of x_m , we get

$$\forall z, \tilde{z} \in \mathbb{U}_\rho, \left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}} \right|(\tilde{z}) \leq \left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}} \right|(x_m) \leq C_1 \left| \overline{\phi_{\epsilon_m}^\beta}^{\omega_i^{\epsilon_m}} \right|(z)$$

which concludes the proof of Step 2.

STEP 3 : There exists $B_{t+1}(\rho) > 0$ such that for all $1 \leq \beta \leq n+1$, either

$$\forall x \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho), \left| \phi_\epsilon^\beta(x) \right| \leq B_{t+1}(\rho)$$

or

$$\forall x, y \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho), \frac{\left| \phi_\epsilon^\beta(y) \right|}{B_{t+1}(\rho)} \leq \left| \phi_\epsilon^\beta(x) \right| \leq B_{t+1}(\rho) \left| \phi_\epsilon^\beta(y) \right|.$$

The proof is the same as in Step 2. Notice that if $m_0(\rho) > 0$, the first inequality holds by Claim 21.

STEP 4 : We prove that there exists $K_i(\rho) > 0$ such that for $0 \leq i \leq t$, and for all $x \in \mathbb{D}_{\tau_{i+1}^\epsilon} \setminus \mathbb{D}_{t_i^\epsilon}$,

$$\left| F_\epsilon \right| (x) \leq K_i(\rho) \left\{ \max_{\partial \mathbb{D}_{t_i^\epsilon}} |F_\epsilon| + \ln \left(\frac{|x|}{t_i^\epsilon} \right) \right\} \quad (4.58)$$

where $t_i^\epsilon = 10R_0\omega_i^\epsilon$, $\tau_{i+1}^\epsilon = \frac{\omega_{i+1}^\epsilon}{10R_0}$ and $F_\epsilon(x) = \widetilde{\Phi}_\epsilon^l(a_\epsilon + x)$.

Let $1 \leq \beta \leq n+1$. We set

$$N_i^\epsilon = \{t_i^\epsilon \leq t \leq \tau_i^\epsilon; \exists x \in \mathbb{R}^2, |x| = t \text{ and } F_\epsilon(x) = 0\}.$$

Then, by the Courant Nodal theorem, N_i^ϵ has a finite number of connected components, bounded by $k+1$, since each connected component adds at least one nodal domain for the eigenfunction Φ_ϵ^β . By Step 1, we clearly have that

$$\forall x \in \mathbb{R}^2; |x| \in N_i^\epsilon \Rightarrow \left| F_\epsilon^\beta(x) \right| \leq A_i(\rho). \quad (4.59)$$

We let

$$c_{i,1}^\epsilon < d_{i,1}^\epsilon < c_{i,2}^\epsilon < d_{i,2}^\epsilon < \dots < c_{i,q_\epsilon}^\epsilon < d_{i,q_\epsilon}^\epsilon$$

be such that

$$N_i^\epsilon = [t_i^\epsilon, \tau_i^\epsilon] \setminus \bigcup_{j=1}^{q_\epsilon} [c_{i,j}^\epsilon, d_{i,j}^\epsilon]$$

with $\{q_\epsilon\}$ a bounded sequence of integers. Let $1 \leq j \leq q_\epsilon$. Then, F_ϵ^β does not vanish on $\mathbb{D}_{d_{i,j}^\epsilon} \setminus \mathbb{D}_{c_{i,j}^\epsilon}$, and we can assume that $F_\epsilon^\beta > 0$ up to take $-F_\epsilon^\beta$. By the eigenvalue equation, F_ϵ^β is then superharmonic on $\mathbb{D}_{d_{i,j}^\epsilon} \setminus \mathbb{D}_{c_{i,j}^\epsilon}$. We set

$$f_\epsilon(t) = \frac{\int_{\partial \mathbb{D}_t} F_\epsilon^\beta(x) d\sigma(x)}{2\pi t}$$

Then,

$$\begin{aligned}
 f'_\epsilon(t) &= \frac{\int_{\partial \mathbb{D}_t} \partial_\nu F_\epsilon^\beta(x) d\sigma(x)}{2\pi t} \\
 &= \frac{-\int_{\mathbb{D}_t} \Delta F_\epsilon^\beta(x) dx}{2\pi t} \\
 &= \frac{-\int_{\mathbb{D}_{c_{i,j}^\epsilon}} \Delta F_\epsilon^\beta(x) dx - \int_{\mathbb{D}_t \setminus \mathbb{D}_{c_{i,j}^\epsilon}} \Delta F_\epsilon^\beta(x) dx}{2\pi t}
 \end{aligned}$$

so that

$$f_\epsilon(t) = f_\epsilon(c_{i,j}^\epsilon) - \frac{\int_{\mathbb{D}_{c_{i,j}^\epsilon}} \Delta F_\epsilon^\beta(x) dx}{2\pi} \ln\left(\frac{t}{c_{i,j}^\epsilon}\right) - \int_{c_{i,j}^\epsilon}^t \frac{\int_{\mathbb{D}_u \setminus \mathbb{D}_{c_{i,j}^\epsilon}} \Delta F_\epsilon^\beta(x) dx}{2\pi u} du.$$

By a Hölder inequality,

$$\left| \int_{\mathbb{D}_{c_{i,j}^\epsilon}} \Delta F_\epsilon^\beta(x) dx \right| \leq \left(\int_M (\phi_\epsilon^\beta)^2 e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}} \left(\int_M e^{2u_\epsilon} dv_g \right)^{\frac{1}{2}} \leq 1$$

and since F_ϵ^β is superharmonic on $\mathbb{D}_{d_{i,j}^\epsilon} \setminus \mathbb{D}_{c_{i,j}^\epsilon}$,

$$f_\epsilon(t) \leq f_\epsilon(c_{i,j}^\epsilon) + \frac{1}{2\pi} \ln\left(\frac{t}{c_{i,j}^\epsilon}\right) \text{ for } c_{i,j}^\epsilon \leq t \leq d_{i,j}^\epsilon.$$

By the second condition of Step 1, we have for $c_{i,j}^\epsilon \leq t \leq d_{i,j}^\epsilon$ that

$$\forall x \in \partial \mathbb{D}_t, F_\epsilon^\beta(x) \leq A_i(\rho) f_\epsilon(t).$$

Gathering these inequalities, for $1 \leq j \leq q_\epsilon$, we get a constant $K_i(\rho) > 0$ such that

$$\forall x \in \partial \mathbb{D}_t, |F_\epsilon^\beta(x)| \leq K_i(\rho) \left(\max_{\partial \mathbb{D}_{t_i}^\epsilon} |F_\epsilon^\beta| + \ln\left(\frac{t}{t_i^\epsilon}\right) \right), \quad (4.60)$$

which is exactly Step 4.

We are now in position to prove the claim. By Step 2, we get some constant $L_i(\rho) > 0$ such that for $1 \leq i \leq t$,

$$\sup_{\mathbb{D}_{t_i}^\epsilon \setminus \mathbb{D}_{\tau_i}^\epsilon} |F_\epsilon| \leq L_i(\rho) \left(\inf_{\mathbb{D}_{t_i}^\epsilon \setminus \mathbb{D}_{\tau_i}^\epsilon} |F_\epsilon| + 1 \right). \quad (4.61)$$

By Step 3, we get some constant $L_{t+1}(\rho)$ such that

$$\sup_{M(\rho)} |\Phi_\epsilon| \leq L_{t+1}(\rho) \left(\max_{\partial \mathbb{D}_{\tau_{t+1}}^\epsilon} |F_\epsilon| + 1 \right). \quad (4.62)$$

By (4.54) in Claim 27,

$$\sup_{\mathbb{D}_{t_0}^\epsilon} |F_\epsilon| \leq C_2 \left(\frac{1}{10R_0} \right). \quad (4.63)$$

Gathering (4.58), (4.61), (4.62) and (4.63), we get the claim. \diamond

In the following claim, we aim at passing to the limit in the equation (i) and the condition (ii) given by proposition 2 at the scale α_ϵ . The limiting function would then satisfy (4.67) and (4.68).

Claim 29. *We have*

— For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x \in \Omega(\rho), |\hat{\Phi}_\epsilon|^2(x) \geq 1 - \beta_\epsilon. \quad (4.64)$$

— For $\rho > 0$ and $x \in \Omega(\rho)$, we set $\hat{\Psi}_\epsilon = \frac{\hat{\Phi}_\epsilon}{|\hat{\Phi}_\epsilon|}$. Then for any $\rho > 0$, $\{\hat{\Psi}_\epsilon\}$ is uniformly equicontinuous on $\mathcal{C}^0(\Omega(\rho), \mathbb{S}^n)$.

— For any $\rho > 0$, up to the extraction of a subsequence of $\{\hat{\Phi}_\epsilon\}$, there exist functions $\hat{\Phi} \in W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}) \cap L^\infty(\Omega(\rho), \mathbb{R}^{n+1})$ and $\hat{\Psi} \in W^{1,2}(\Omega(\rho), \mathbb{S}^n) \cap \mathcal{C}^0(\Omega(\rho), \mathbb{S}^n)$ such that

$$\hat{\Phi}_\epsilon \rightharpoonup \hat{\Phi} \text{ in } W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}) \quad (4.65)$$

and

$$\hat{\Psi}_\epsilon \rightarrow \hat{\Psi} \text{ in } \mathcal{C}^0(\Omega(\rho), \mathbb{S}^n) \text{ as } \epsilon \rightarrow 0 \quad (4.66)$$

with

$$|\hat{\Phi}|^2 \geq_{a.e.} 1 \text{ and } \hat{\Psi} = \frac{\hat{\Phi}}{|\hat{\Phi}|} \quad (4.67)$$

and for $1 \leq i \leq n+1$,

$$\Delta \hat{\phi}^i = \Lambda_k(M, [g]) \hat{\psi}^i d\hat{v} \quad (4.68)$$

in a weak sense on $\Omega(\rho)$.

Proof.

STEP 1 : We recall that $a_\epsilon \rightarrow a$ as $\epsilon \rightarrow 0$ with $\widetilde{z_{i_0}} = a$.

For $1 \leq j \leq s_0$ and $\theta_\epsilon = \frac{\epsilon}{e^{2\bar{v}_1(a)} \alpha_\epsilon^2}$,

$$\sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz = O(e^{-\frac{\rho^2}{8\theta_\epsilon}}). \quad (4.69)$$

For $0 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\bar{v}_1(a)} (\omega_i^\epsilon)^2}$,

$$\sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} |\overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z)|^2 \overline{p}_\epsilon^{\omega_i^\epsilon}\left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} x\right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}). \quad (4.70)$$

For $1 \leq i \leq s$ and $i \neq i_0$,

$$\sup_{x \in \Omega(\rho)} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(z)|^2 p_\epsilon(\check{x}, z) dv_g(z) = O(e^{-\frac{\rho^2}{8\epsilon}}). \quad (4.71)$$

Note that (4.71) was already proved in Step 1 of Claim 22. Note also that the proof of (4.69) reduces to (4.70) for $i = 0$. Let $0 \leq i \leq t$ and $1 \leq j \leq s_i$. Then, for $y \in \Omega(\rho)$,

$$\begin{aligned} e^{\frac{\rho^2}{8\tau_i^\epsilon}} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z) \right|^2 \overline{p}_\epsilon^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} y \right) dz \\ \leq \frac{\int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z) \right|^2 e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}}}{\inf_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}}} \times O \left(\frac{e^{-\frac{\rho^2}{4\tau_i^\epsilon} \left(\frac{9}{10^2} - \frac{1}{2} - \frac{1}{100} \right)}}{\tau_i^\epsilon} \right) \\ \leq \frac{C_0}{\inf_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}}} \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\tau_i^\epsilon} \end{aligned}$$

where we used the uniform bound (4.7) on $\overline{p}_\epsilon^{\omega_i^\epsilon}$ on $\mathbb{D}_{\frac{1}{\rho}} \times \mathbb{D}_{\frac{1}{\rho}}$. We assume by contradiction that

$$\inf_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}} \leq \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\tau_i^\epsilon}.$$

Let $y \in M$ be such that $\overline{y}^{\omega_i^\epsilon} \in \mathbb{D}_{\frac{\rho}{10}}(p_{i,j})$. Then,

$$\begin{aligned} e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}(\overline{y}^{\omega_i^\epsilon})} &= e^{2v_l(y)} (\omega_i^\epsilon)^2 \int_M p_\epsilon(x, y) d\nu_\epsilon(y) \\ &\geq \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} \overline{p}_\epsilon^{\omega_i^\epsilon}(z, \overline{y}^{\omega_i^\epsilon}) d\overline{\nu}_\epsilon^{\omega_i^\epsilon}(z) \\ &\geq \alpha_0 \frac{e^{-\frac{\rho^2}{80\tau_i^\epsilon}}}{\tau_i^\epsilon} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} d\overline{\nu}_\epsilon^{\omega_i^\epsilon}. \end{aligned}$$

For $z \in \mathbb{D}_{\frac{\rho}{20}}(p_{i,j})$,

$$\begin{aligned} e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}(z)} &\leq \frac{A_0 \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} d\overline{\nu}_\epsilon^{\omega_i^\epsilon} + O \left(e^{-\frac{\rho^2}{4\tau_i^\epsilon} \left(\frac{1}{20^2} - \frac{1}{1000} \right)} \right)}{4\pi\tau_i^\epsilon} \\ &\leq \frac{A_0}{4\pi\alpha_0} \frac{e^{-\frac{\rho^2}{16\tau_i^\epsilon}} + O \left(e^{-\frac{3\rho^2}{8000\tau_i^\epsilon}} \right)}{\tau_i^\epsilon}. \end{aligned}$$

Then, $e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}} \rightarrow 0$ uniformly on $\mathcal{C}^0(\mathbb{D}_{\frac{\rho}{20}}(p_{i,j}))$ as $\epsilon \rightarrow 0$ and $\lambda_\star(\mathbb{D}_{\frac{\rho}{20}}(p_{i,j}), e^{2\overline{u}_\epsilon^{\omega_i^\epsilon}} \xi) \rightarrow +\infty$ as $\epsilon \rightarrow 0$ which contradicts (4.48) in Claim 26. This concludes the proof of Step 1.

STEP 2 : There exists a sequence $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x, y \in \Omega(\rho), |x - y| \leq \frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon} \Rightarrow |\Phi_\epsilon(x) - \Phi_\epsilon(y)| \leq \beta_\epsilon. \quad (4.72)$$

We set $\gamma_\epsilon = \|\theta_\epsilon e^{2\hat{u}_\epsilon}\|_{C^0(\Omega(\rho))}^{\frac{1}{3}}$. We have that $\gamma_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. Indeed, for $r > 0$, and $x \in \Omega(\rho)$,

$$\theta_\epsilon e^{2\hat{u}_\epsilon(x)} \leq \left(\frac{A_0}{4\pi} + o(1) \right) \int_{\mathbb{D}_r(x)} d\hat{v}_\epsilon + o(1) \leq \frac{A_0 \hat{v}(\mathbb{D}_r(x))}{4\pi} + o(1) \leq \frac{A_0 D_1(\rho)}{4\pi \ln(\frac{1}{r})} + o(1)$$

since we have (4.45) and thanks successively to (4.8) (4.7) and to (4.46), (4.55). We also have $\frac{\gamma_\epsilon}{\sqrt{\theta_\epsilon}} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ since $\frac{\theta_\epsilon^{\frac{1}{3}}}{\gamma_\epsilon} = \|e^{2\hat{u}_\epsilon}\|_{C^0(\Omega(\rho))}^{-\frac{1}{3}} \leq m_i(\rho)^{-\frac{1}{3}}$ is bounded and we have (4.45). Let x_ϵ and $y_\epsilon \in \Omega(\rho)$ with $|x_\epsilon - y_\epsilon| \leq \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon}$. We set

$$F_\epsilon(z) = \hat{\Phi}_\epsilon(x_\epsilon + \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon} z)$$

and we let α_ϵ be the mean value of F_ϵ in \mathbb{D}_3 . Then, by Poincaré and Sobolev inequalities, we get a constant $K > 0$ such that

$$\begin{aligned} \|F_\epsilon - \alpha_\epsilon\|_{L^\infty(\mathbb{D}_2(0))} &\leq K \|\Delta F_\epsilon\|_{L^\infty(\mathbb{D}_3(0))} + K \|\nabla F_\epsilon\|_{L^2(\mathbb{D}_3(0))} \\ &\leq K \|\hat{\Phi}_\epsilon\|_{L^\infty(\Omega(\rho))} \lambda_\epsilon \gamma_\epsilon + K \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}} \\ &\leq KC_2(\rho) \lambda_\epsilon \gamma_\epsilon + K \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}} \end{aligned}$$

thanks successively to (4.56) and (4.54). Setting

$$\beta_\epsilon = 2KC_2(\rho) \lambda_\epsilon \gamma_\epsilon + 2K \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}},$$

$\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and we get Step 2.

STEP 3 : There exists a sequence $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that for all $x \in M$,

$$\hat{x} \in \Omega(\rho) \Rightarrow \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon \quad (4.73)$$

and

$$\hat{x} \in \Omega(\rho) \cap \text{supp}(\hat{v}_\epsilon) \Rightarrow |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon \quad (4.74)$$

Note that (4.73) gives (4.64) for $x \in \text{supp}(v_\epsilon)$ by Proposition 2. Let $x \in M$ be such that

$\hat{x} \in \Omega(\rho)$.

$$\begin{aligned}
 \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| &\leq \int_M p_\epsilon(x, y) \left| |\Phi_\epsilon(x)|^2 - |\Phi_\epsilon(y)|^2 \right| dv_g(y) \\
 &\leq \int_{\mathbb{D}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - |\hat{\Phi}_\epsilon(z)|^2 \right| dz \\
 &\quad + I_\epsilon \\
 &\quad + \sum_{i \neq i_0} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2 p_\epsilon(x, y) dv_g(y) \\
 &\quad + \sum_{i=0}^t \sum_{j=1}^{s_t} \int_{\mathbb{D}_\rho(p_{i,j})} \left| \overline{\Phi_\epsilon}^{\omega_i^\epsilon} \right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} \hat{x} \right) dz,
 \end{aligned}$$

where

$$I_\epsilon = \int_{M \setminus \mathbb{D}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} p_\epsilon(x, y) \left(C_2^2(\rho) + C_0^2(\rho) \left(\ln \left(1 + \frac{d_g(y, \bar{a}_\epsilon)}{\alpha_\epsilon} \right) + 1 \right)^2 \right) dv_g(y).$$

Here, we used Claim 27 and Claim 28. By (4.69), (4.70), (4.71) and (4.72),

$$\left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq 2C_2(\rho)\beta_\epsilon + O(e^{-\frac{\rho^2}{8\alpha_\epsilon}}) + I_\epsilon$$

and there exists some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that

$$\begin{aligned}
 I_\epsilon &\leq K_0(\rho) \ln \left(\frac{\delta(M)}{\alpha_\epsilon} \right)^2 \int_{M \setminus \Omega_l} p_\epsilon(x, y) dv_g(y) \\
 &\quad + K_1(\rho) \int_{\hat{\Omega}_l \setminus \mathbb{D}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) (\ln(1 + |z|)^2 + 1) dz.
 \end{aligned}$$

Since $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$,

$$\ln \left(\frac{\delta(M)}{\alpha_\epsilon} \right)^2 \int_{M \setminus \Omega_l} p_\epsilon(x, y) dv_g(y) \leq \ln \left(\frac{\delta(M)}{\alpha_\epsilon} \right)^2 \times O \left(\frac{e^{-\frac{\delta(M)^2}{4\epsilon}}}{\epsilon} \right) = o(1) \text{ as } \epsilon \rightarrow 0$$

and by (4.5),

$$\begin{aligned}
 &\int_{\hat{\Omega}_l \setminus \mathbb{D}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) (\ln(1 + |z|)^2 + 1) dz \\
 &\leq \int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \frac{A_0}{4\pi\theta_\epsilon} e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}} (\ln(1 + |z|)^2 + 1) dz \\
 &\leq \int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{1}{\beta_\epsilon}}(0)} \frac{A_0}{4\pi} e^{-\frac{|y|^2}{8}} \left(\ln \left(1 + \left| \hat{x} + \sqrt{\theta_\epsilon} y \right| \right)^2 + 1 \right) dy \\
 &= o(1) \text{ uniformly for } \hat{x} \in \Omega(\rho).
 \end{aligned}$$

Up to increase β_ϵ , we get (4.73). The same estimates can be obtained for $|\Phi_\epsilon|$ instead of $|\Phi_\epsilon|^2$, and we get up to increase β_ϵ for $x \in M$ such that $\hat{x} \in \Omega(\rho)$,

$$||\hat{\Phi}_\epsilon(\hat{x})| - K_\epsilon[|\Phi_\epsilon|](x)| \leq \beta_\epsilon.$$

Since, if $z \in \text{supp}(\hat{v}_\epsilon) \cap \Omega(\rho)$, we have

$$||\hat{\Phi}_\epsilon(z)|^2 - 1| \leq \beta_\epsilon,$$

up to increase β_ϵ , we get for $x \in M$ such that $\hat{x} \in \text{supp}(\hat{v}_\epsilon) \cap \Omega(\rho)$,

$$|K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon.$$

We follow the proof of Claim 9 in [91] to prove that $\hat{\Psi}_\epsilon$ is uniformly equicontinuous on $\Omega(\rho)$. Indeed, we can use the Poincaré inequality of Theorem 11 thanks to Claim 26. Therefore, up to the extraction of a subsequence, $\hat{\Psi}_\epsilon \rightarrow \hat{\Psi}$ in $\mathcal{C}^0(\Omega(\rho), \mathbb{S}^n)$ as $\epsilon \rightarrow 0$.

STEP 4 : We have that

$$\hat{\phi}_\epsilon^i e^{2\hat{u}_\epsilon} dx \rightarrow_\star \hat{\psi}^i \hat{v} \text{ in } \mathcal{M}(\Omega(\rho)) \text{ as } \epsilon \rightarrow 0.$$

Let $\zeta \in \mathcal{C}_c^0(\Omega(\rho))$ and $R > \frac{1}{\rho}$. Then

$$\begin{aligned} & \int_{\mathbb{R}^2} \zeta(z) \left(\hat{\phi}_\epsilon^i(z) e^{2\hat{u}_\epsilon(z)} dz - \hat{\psi}^i(z) d\hat{v}(z) \right) \\ &= \int_{M \setminus \mathbb{D}_R} \left(\int_{\hat{\Omega}(\rho)} p_\epsilon(x, y) \zeta(y) \phi_\epsilon^i(y) dv_g(y) \right) dv_\epsilon(x) \\ &+ \int_{\mathbb{D}_R} \left(\int_{\mathbb{D}_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ &+ \int_{\Omega(\rho)} \zeta(x) \left(\int_{\mathbb{D}_R} (\hat{\psi}_\epsilon^i(z) - \hat{\psi}_\epsilon^i(x)) |\hat{\Phi}_\epsilon(z)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ &+ \int_{\Omega(\rho)} \zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{\mathbb{D}_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ &+ \int_{\Omega(\rho)} \left(\zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) - \zeta(x) \hat{\psi}^i(x) d\hat{v}(x) \right). \end{aligned}$$

We have by (4.9) that

$$\begin{aligned} & \int_{M \setminus \mathbb{D}_R} \left(\int_{\hat{\Omega}(\rho)} p_\epsilon(x, y) \zeta(y) \phi_\epsilon^i(y) dv_g(y) \right) dv_\epsilon(x) \\ & \leq \|\zeta\|_\infty C_2(\rho) \sup_{y \in M \setminus \mathbb{D}_R} \int_{\mathbb{D}_{\frac{1}{\rho}}} p_\epsilon(x, y) dv_g(x) \\ & = o(1) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

By Step 1, Claim 28 and (4.7),

$$\begin{aligned}
& \int_{\mathbb{D}_R} \left(\int_{\mathbb{D}_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
& \leq \sup_{x \in \mathbb{D}_R} \int_{\mathbb{D}_R} |\zeta(z) - \zeta(x)| \left| \hat{\phi}_\epsilon^i(z) \right| \hat{p}_\epsilon(z, x) dz \\
& \leq \sum_{j=1}^{s_0} \sup_{x \in \mathbb{D}_R} |\zeta(x)| \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} \left| \hat{\phi}_\epsilon^i(z) \right| \hat{p}_\epsilon(z, x) dz \\
& \quad + \sup_{x \in \mathbb{D}_R} \int_{\mathbb{D}_R \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} |\zeta(z) - \zeta(x)| \left| \hat{\phi}_\epsilon^i(z) \right| \hat{p}_\epsilon(z, x) dz \\
& \leq \|\zeta\|_\infty \sum_{j=1}^{s_0} \sup_{x \in \Omega(\rho)} \left(\int_{\mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz \right)^{\frac{1}{2}} \\
& \quad + C_0(\rho) (1 + \ln(1 + C_0 R)) \sup_{x \in \mathbb{D}_R} \int_{\mathbb{R}^2} |\zeta(z) - \zeta(x)| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{2\pi\theta_\epsilon} dz \\
& = o(1) \text{ as } \epsilon \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega(\rho)} \zeta(x) \left(\int_{\mathbb{D}_R} (\hat{\psi}_\epsilon^i(z) - \hat{\psi}_\epsilon^i(x)) |\hat{\Phi}_\epsilon(z)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
& \leq 2 \|\zeta\|_\infty \sum_{j=1}^{s_0} \sup_{x \in \Omega(\rho)} \left(\int_{\mathbb{D}_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz \right)^{\frac{1}{2}} \\
& \quad + \|\zeta\|_\infty C_2 \left(\frac{\rho}{10} \right) \sup_{x \in \Omega(\rho)} \int_{\Omega(\frac{\rho}{10})} |\hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z)| \hat{p}_\epsilon(z, x) dz \\
& \quad + 2 \|\zeta\|_\infty C_0(\rho) (1 + \ln(1 + C_0 R)) \sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_R \setminus \Omega(\frac{\rho}{10})} \hat{p}_\epsilon(z, x) dz \\
& = o(1) \text{ as } \epsilon \rightarrow 0,
\end{aligned}$$

where by (4.7),

$$\begin{aligned}
& \sup_{x \in \Omega(\rho)} \int_{\Omega(\frac{\rho}{10})} |\hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z)| \hat{p}_\epsilon(z, x) dz \\
& \leq \sup_{x \in \Omega(\rho)} \int_{\Omega(\frac{\rho}{10})} |\hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z)| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{2\pi\theta_\epsilon} dz \\
& = o(1) \text{ as } \epsilon \rightarrow 0.
\end{aligned}$$

We also have that

$$\begin{aligned} \int_{\Omega(\rho)} \zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{\mathbb{D}_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ \leq \|\zeta\|_\infty \sup_{x \in \Omega(\rho) \cap \text{supp}(\hat{v}_\epsilon)} \int_{\mathbb{D}_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz . \end{aligned}$$

We use (4.74) of Step 3, in order to obtain that

$$\sup_{x \in \Omega(\rho) \cap \text{supp}(\hat{v}_\epsilon)} \int_{\mathbb{D}_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \rightarrow 0 \text{ as } \epsilon \rightarrow 0 . \quad (4.75)$$

Let $x \in M$ be such that $\hat{x} \in \Omega(\rho) \cap \text{supp}(\hat{v}_\epsilon)$,

$$\begin{aligned} K_\epsilon[|\Phi_\epsilon|](x) - 1 &= \int_{M \setminus \mathbb{D}_R} (|\Phi_\epsilon(y)| - 1) p_\epsilon(x, y) dv_g(y) \\ &\quad + \int_{\mathbb{D}_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, \hat{x}) dz \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{M \setminus \mathbb{D}_R} (|\Phi_\epsilon(y)| - 1) p_\epsilon(x, y) dv_g(y) \right| \\ & \leq \int_{M \setminus \Omega_l} p_\epsilon(x, y) dv_g(y) K_0(\rho) \ln \left(\frac{\delta(M)}{\alpha_\epsilon} \right) \\ & \quad + K_1(\rho) \int_{\hat{\Omega}_l \setminus \mathbb{D}_R} \hat{p}_\epsilon(z, \hat{x}) (1 + \ln(1 + |z|)) dz \\ & \leq O \left(\frac{e^{-\frac{\delta(M)^2}{4\epsilon}}}{4\pi\epsilon} \ln \left(\frac{\delta(M)}{\alpha_\epsilon} \right) \right) \\ & \quad + K_1(\rho) \int_{\mathbb{R}^2 \setminus \mathbb{D}_R} A_0 \frac{e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}}}{4\pi\theta_\epsilon} (1 + \ln(1 + |z|)) dz \\ & \leq O \left(\int_{\mathbb{R}^2 \setminus \mathbb{D}_{\frac{R}{\sqrt{\theta_\epsilon}}}} e^{-\frac{|y|^2}{8}} (1 + \ln(1 + |\hat{x} + \sqrt{\theta_\epsilon} y|)) dz \right) \\ & = o(1) \text{ as } \epsilon \rightarrow 0 . \end{aligned}$$

This gives (4.75). By (4.10),

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \sup_{x \in \mathbb{D}_{\frac{1}{\hat{\rho}}}} \left| \int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0 ,$$

so that

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\Omega(\rho)} \left(\zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{\mathbb{D}_R} \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) - \zeta(x) \hat{\psi}^i(x) d\hat{v}(x) \right) \right) = 0 .$$

Gathering all these computations, we get Step 4.

As a conclusion, (4.73) in Step 3 gives (4.64) for $x \in \text{supp}(\nu_\epsilon)$ by Proposition 2. In the remark before Step 4, we get (4.66). Then, (4.64), (4.65) and (4.66) give (4.67). We finally get (4.68) passing to the limit in the equation satisfied by $\hat{\phi}_\epsilon^i$ thanks to Step 4. This ends the proof of the Claim. \diamond

Thanks to Claim 29, a diagonal extraction gives some functions $\hat{\Phi} : \mathbb{R}^2 \setminus \{p_{0,1}, \dots, p_{0,s_0}\} \rightarrow \mathbb{R}^{n+1}$ and $\hat{\Psi} : \mathbb{R}^2 \setminus \{p_{0,1}, \dots, p_{0,s_0}\} \rightarrow \mathbb{S}^n$ such that for any $\rho > 0$, the conclusions (4.65) (4.66) (4.67) and (4.68) of Claim 29 hold true for $\hat{\Phi}$ and $\hat{\Psi}$. We denote by ν the measure without atom such that

$$e^{2\hat{u}_\epsilon} dx \rightharpoonup_\star \nu \text{ in } \mathcal{M}(\Omega(\rho)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$. (Notice that, $\nu = \hat{\nu}$ on $\mathbb{R}^2 \setminus \{p_{0,1}, \dots, p_{0,s_0}\}$)

Claim 30.

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{\Omega(\rho)} |\nabla \hat{\Phi}_\epsilon|^2 dx \geq \int_{\mathbb{R}^2} \frac{|\nabla \hat{\Phi}|^2}{|\hat{\Phi}|} dx \geq \Lambda_k(M, [g]) \int_{\mathbb{R}^2} d\nu + \int_{\mathbb{R}^2} \frac{|\nabla |\hat{\Phi}||^2}{|\hat{\Phi}|} dx \quad (4.76)$$

where $\int_{\mathbb{R}^2} d\nu = \lim_{\rho \rightarrow 0} m_i(\rho) \geq m_i$.

Proof.

Let $\eta \in \mathcal{C}_c^\infty(\Omega(\sqrt{\rho}))$ be given by Claim 18 with $\eta \geq 1$ on $\Omega(\rho)$ and

$$\int_{\mathbb{R}^2} |\nabla \eta|^2 \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

Integrating against $\psi^i \eta$ the equation (4.68) and summing over i give that

$$\Lambda_k(M, [g]) \int_{\mathbb{R}^2} \eta d\hat{\nu} = \int_{\mathbb{R}^2} \langle \nabla \eta, \nabla |\hat{\Phi}| \rangle dx + \int_{\mathbb{R}^2} \left(\frac{|\nabla \hat{\Phi}|^2}{|\hat{\Phi}|^2} - \frac{|\nabla |\hat{\Phi}||^2}{|\hat{\Phi}|^2} \right) \eta dx.$$

Since $\hat{\Phi}_\epsilon \rightharpoonup \hat{\Phi}$ in $W^{1,2}(M(\rho), \mathbb{R}^{n+1})$ and $|\hat{\Phi}| \geq_{a.e.} 1$ we get

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega(\rho)} |\nabla \hat{\Phi}_\epsilon|^2 dx &\geq \int_{\Omega(\rho)} |\nabla \hat{\Phi}|^2 dx \\ &\geq \int_{\mathbb{R}^2} \eta \frac{|\nabla \hat{\Phi}|^2}{|\hat{\Phi}|} dx \\ &\geq \Lambda_k(M, [g]) \int_{\mathbb{R}^2} \eta d\hat{\nu} - \int_{\mathbb{R}^2} \langle \nabla \eta, \nabla |\hat{\Phi}| \rangle dx \\ &\quad + \int_{\mathbb{R}^2} \frac{|\nabla |\hat{\Phi}||^2}{|\hat{\Phi}|} \eta dx \\ &\geq \Lambda_k(M, [g]) m_i(\sqrt{\rho}) - C' \sqrt{\frac{C}{\ln\left(\frac{1}{\rho}\right)}} + \int_{\Omega(\sqrt{\rho})} \frac{|\nabla |\hat{\Phi}||^2}{|\hat{\Phi}|} dx \end{aligned}$$

where C and C' are some constants independent of ρ . Passing to the limit as $\rho \rightarrow 0$, we get (4.76). Thanks to (4.35), (4.37) and (4.52), we finally get the claim. \diamond

4.6.2 Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$

We now assume that $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$, we let $\theta_0 = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{e^{2v_l(a)} \alpha_\epsilon^2}$ and we denote by \hat{v} the weak^{*} limit of \hat{v}_ϵ in $\mathcal{M}(\mathbb{R}^2)$. Let $R_0 > 0$ and $x \in \mathbb{D}_{R_0}$. We have by (4.7) that

$$\begin{aligned} e^{2\hat{u}_\epsilon}(x) &= e^{2v_l(\check{x})} \alpha_\epsilon^2 \int_M p_\epsilon(\check{x}, y) d\nu_\epsilon(y) \\ &\leq \frac{A_0 e^{2v_l(\check{x})} \alpha_\epsilon^2}{4\pi\epsilon} \int_M d\nu_\epsilon \\ &\leq \frac{A_0}{4\pi\theta_0} (1 + o(1)). \end{aligned}$$

Since $m_i > 0$, we get that $\theta_0 < +\infty$. Now, we let $e^{2\hat{u}}$ be a smooth function on \mathbb{R}^2 defined by

$$e^{2\hat{u}(x)} = \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{v}(y). \quad (4.77)$$

Let $R_0 > 0$, $R > R_0$ and $x \in \mathbb{D}_{R_0}$. We have that

$$\begin{aligned} \left| e^{2\hat{u}_\epsilon}(x) - e^{2\hat{u}(x)} \right| &= \left| \int_M \alpha_\epsilon^2 p_\epsilon(\check{x}, y) d\nu_\epsilon(y) - e^{2\hat{u}(x)} \right| \\ &\leq \int_{M \setminus \mathbb{D}_R} \alpha_\epsilon^2 p_\epsilon(\check{x}, y) d\nu_\epsilon(y) \\ &\quad + \left| \int_{\mathbb{D}_R} \hat{p}_\epsilon(x, y) d\hat{v}_\epsilon(y) - \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{v}(y) \right| \\ &\leq o(1) + \frac{A_0}{4\pi\theta_0} (1 + o(1)) e^{-\frac{(R-R_0)^2}{8\theta_0}} \\ &\quad + \left| \int_{\mathbb{D}_R} \left(\hat{p}_\epsilon(x, y) - \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} \right) d\hat{v}_\epsilon \right| \\ &\quad + \left| \int_{\mathbb{D}_R} \frac{e^{-\frac{|x-y|^2}{8\theta_0}}}{4\pi\theta_0} (d\hat{v}_\epsilon - d\hat{v}) \right| + \int_{\mathbb{R}^2 \setminus \mathbb{D}_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{v} \\ &\rightarrow \frac{A_0}{4\pi\theta_0} e^{-\frac{(R-R_0)^2}{8\theta_0}} + \int_{\mathbb{R}^2 \setminus \mathbb{D}_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} d\hat{v} \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Letting $R \rightarrow +\infty$, we get for any $R_0 > 0$ that

$$e^{2\hat{u}_\epsilon} \rightarrow e^{2\hat{u}} \text{ in } \mathcal{C}^0(\mathbb{D}_{R_0}) \text{ as } \epsilon \rightarrow 0. \quad (4.78)$$

With Claim 19, $\{\hat{\phi}_\epsilon^i\}$ is bounded in $L^2(\mathbb{D}_R)$ for any $R > 0$. With (4.78) and elliptic estimates on the equation

$$\Delta \hat{\phi}_\epsilon^i = \lambda_\epsilon e^{2\hat{u}_\epsilon} \hat{\phi}_\epsilon^i,$$

we get some smooth function $\hat{\Phi}$ on \mathbb{R}^2 such that for any $R_0 > 0$

$$\hat{\phi}_\epsilon^i \rightarrow \hat{\phi}^i \text{ in } \mathcal{C}^1(\mathbb{D}_{R_0}) \text{ as } \epsilon \rightarrow 0. \quad (4.79)$$

and

$$\Delta \hat{\phi}^i = \Lambda_k(M, [g]) e^{2\hat{u}} \hat{\phi}^i \text{ in } \mathbb{R}^2. \quad (4.80)$$

We now prove the following :

Claim 31. *We have the energy inequality*

$$\int_{\mathbb{R}^2} |\nabla \hat{\Phi}(x)|^2 dx \geq \Lambda_k(M, [g]) \int_{\mathbb{R}^2} e^{\hat{u}}. \quad (4.81)$$

Proof.

STEP 1 : Up to the extraction of a subsequence, there exists some sequences $\{\omega_i^\epsilon\}$ with $0 \leq i \leq t+1$ and $0 \leq t \leq k$ and

$$\alpha_\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \dots \ll \omega_{t+1}^\epsilon = \delta_0$$

and for $1 \leq i \leq t$ and $1 \leq j \leq s_i$ some points $p_{i,j} \in \mathbb{A}_{\frac{1}{R_0}}$ with $R_0 > 0$ and $s-1 + \sum_{i=1}^t s_i \leq k$ such that for all $\rho > 0$, there exists $C_0(\rho)$ such that

$$\begin{aligned} \forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i, \rho) \cup \bigcup_{i=1}^t \bigcup_{j=1}^{s_i} \Omega_{i,j} \right), \\ |\Phi_\epsilon|(x) \leq C_0(\rho) \left(\ln \left(1 + \frac{d_g(\bar{a}_\epsilon, x)}{\sqrt{\epsilon}} \right) + 1 \right) \end{aligned}$$

where $\widetilde{\Omega}_{i,j} = \omega_i^\epsilon \mathbb{D}_\rho(p_{i,j}) + a_\epsilon$ and $\bar{a}_\epsilon = \exp_{g_l, x_l}^{-1}(a_\epsilon)$. We also have that for all $\rho > 0$,

$$\sup_{x \in \Omega(\rho)} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi_\epsilon^{\omega_i^\epsilon}}(z) \right|^2 \overline{p_\epsilon^{\omega_i^\epsilon}} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} x \right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}). \quad (4.82)$$

for $1 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\bar{v}_l(a)}(\omega_i^\epsilon)^2}$ and

$$\sup_{x \in \Omega(\rho)} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(z)|^2 p_\epsilon(\check{x}, z) dz = O(e^{-\frac{\rho^2}{8\epsilon}}). \quad (4.83)$$

For $1 \leq i \leq s$ and $i \neq i_0$.

For the estimate of Φ_ϵ , we follow the proof of Claim 26 and Claim 28, using (4.78) and (4.79) instead of the estimates of Claim 27. The proof of (4.82) and (4.83) follows the proof of Step 1 in Claim 29, which is a consequence of Claim 26.

STEP 2 : We have that

$$\int_{\mathbb{R}^2} |\hat{\Phi}(y)|^2 \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{4\pi\theta_0} dy \geq 1. \quad (4.84)$$

In order to prove (4.84), it is sufficient to use Proposition 2 and to prove that for $R_0 > 0$ fixed, $x \in M$ such that $\hat{x} \in \mathbb{D}_{R_0}$, there holds

$$\int_{\mathbb{R}^2} |\hat{\Phi}(y)|^2 \frac{e^{-\frac{|\hat{x}-y|^2}{4\theta_0}}}{4\pi\theta_0} - K_\epsilon[|\Phi_\epsilon|^2](x) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (4.85)$$

Let's prove (4.85). We fix $r > 0$ and $R > r$. Let $x \in M$ be such that $\hat{x} \in \mathbb{D}_r$. We fix $\rho > 0$. Then,

$$\begin{aligned} \left| K_\epsilon[|\Phi_\epsilon|^2](x) - \int_{\mathbb{D}_R} |\hat{\Phi}(z)|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{4\pi\theta_0} dz \right| &= \int_{M \setminus \mathbb{D}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) dv_g(y) \\ &\quad + \int_{\mathbb{D}_R} \hat{p}_\epsilon(z, \hat{x}) |\hat{\Phi}_\epsilon(z)|^2 dz \\ &\quad - \int_{\mathbb{D}_R} |\hat{\Phi}(z)|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{4\pi\theta_0} dz. \end{aligned}$$

There exist some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that, by Step 1,

$$\begin{aligned} \int_{M \setminus \mathbb{D}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) dv_g(y) &\leq K_0(\rho) \int_{M \setminus \Omega_l} \ln \left(\frac{\delta(M)}{\sqrt{\epsilon}} \right)^2 p_\epsilon(x, y) dv_g(y) \\ &\quad + K_1(\rho) \int_{\hat{\Omega}_l \setminus \mathbb{D}_R} (\ln(1 + |z|)^2 + 1) \hat{p}_\epsilon(z, \hat{x}) dz \\ &\quad + \sum_{i=1}^t \sum_{j=1}^{s_i} \int_{\mathbb{D}_{\frac{\rho}{10}}(p_{i,j})} |\overline{\Phi_\epsilon^{\omega_i^\epsilon}}(z)|^2 \overline{p_\epsilon^{\omega_i^\epsilon}} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} \hat{x} \right) dz \\ &\quad + \sum_{i \neq i_0} \int_{B_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) dv_g(y) \\ &\leq O \left(\ln \left(\frac{\delta(M)}{\sqrt{\epsilon}} \right)^2 \frac{e^{-\frac{\delta(M)^2}{4\epsilon}}}{\epsilon} \right) \\ &\quad + O \left(e^{-\frac{\rho^2}{8\tau_1^\epsilon}} \right) \\ &\quad + \frac{K_1(\rho) A_0}{2\pi\theta_0} \int_{\mathbb{R}^2 \setminus \mathbb{D}_R} (\ln(1 + |z|)^2 + 1) e^{-\frac{|\hat{x}-z|^2}{8\theta_0}} dz. \end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$ and then as $R \rightarrow +\infty$, we get (4.85) and then (4.84). This ends the proof of Step 2.

STEP 3 : We have that

$$\Lambda_k(M, [g]) \int_{\mathbb{R}^2} |\hat{\Phi}(y)|^2 e^{2\hat{u}(y)} dy \leq \int_{\mathbb{R}^2} |\nabla \hat{\Phi}(x)|^2 dx . \quad (4.86)$$

By contradiction, we assume that there is $\epsilon_0 > 0$ such that

$$\Lambda_k(M, [g]) \int_{\mathbb{R}^2} |\hat{\Phi}(y)|^2 e^{2\hat{u}(y)} dy \geq \int_{\mathbb{R}^2} |\nabla \hat{\Phi}(x)|^2 dx + \epsilon_0 .$$

We fix $R > 0$. By the equation (4.80),

$$\frac{1}{2} \Delta |\hat{\Phi}|^2 = \Lambda_k(M, [g]) e^{2\hat{u}} |\hat{\Phi}|^2 - |\nabla \hat{\Phi}|^2 .$$

We integrate on \mathbb{D}_R ,

$$-\frac{1}{2} \int_{\partial \mathbb{D}_R} \partial_\nu (|\hat{\Phi}|^2) d\sigma = \int_{\mathbb{D}_R} \left(\Lambda_k(M, [g]) e^{2\hat{u}} |\hat{\Phi}|^2 - |\nabla \hat{\Phi}|^2 \right) \geq \frac{\epsilon_0}{2}$$

for any $R > R_0$, for some $R_0 > 0$, since $\Lambda_k(M, [g]) e^{2\hat{u}} |\hat{\Phi}|^2 - |\nabla \hat{\Phi}|^2 \in L^1(\mathbb{R}^2)$. We set

$$f(r) = \frac{\int_{\partial \mathbb{D}_r} |\hat{\Phi}|^2 d\sigma}{2\pi r} .$$

Then, for $R > R_0$, $2\pi f'(R) \leq -\frac{\epsilon_0}{R}$ so that

$$f(R) \leq -\frac{\epsilon_0}{2\pi} \ln \frac{R}{R_0} + f(R_0) \rightarrow -\infty \text{ as } R \rightarrow +\infty$$

which contradicts the fact that $f(R) > 0$. This ends the proof of Step 3.

We are now in position to get the claim. We integrate (4.84) against \hat{v} and (4.77) against dx , and we obtain

$$\int_{\mathbb{R}^2} |\hat{\Phi}(y)|^2 e^{2\hat{u}(y)} dy \geq \int_{\mathbb{R}^2} d\hat{v} = \int_{\mathbb{R}^2} e^{2\hat{u}(y)} dy \quad (4.87)$$

and we get (4.81) with (4.87) and (4.86).

◇

4.7 Proof of Theorem 9

4.7.1 Regularity of the limiting measures

In this subsection, we aim at proving the following no neck energy and regularity result, keeping the notations of Proposition 3.

Proposition 4. *For $i \in \{1, \dots, N\}$, there exists $q_{i,1}, \dots, q_{i,s_i} \in \mathbb{S}^2$ and $e^{2\check{u}_i} \in L^\infty(\mathbb{S}^2)$, smooth except maybe at one point, positive except maybe at a finite set of points (which correspond to conical singularities of the metric $e^{2\check{u}_i} h$ on \mathbb{S}^2 such that for all $\rho > 0$,*

$$e^{2\check{u}_i^\epsilon} dv_h \rightharpoonup_\star e^{2\check{u}_i} dv_h \text{ on } \mathcal{M}(S_i(\rho)) \text{ as } \epsilon \rightarrow 0$$

with $S_i(\rho) = \mathbb{S}^2 \setminus \left(B_h(p, \rho) \cup \bigcup_{j=1}^{s_i} B_h(q_{i,j}, \rho) \right)$ and $\int_{\mathbb{S}^2} e^{2\tilde{u}_i} dv_h = m_i$.

If $m_0 > 0$, there exists p_1, \dots, p_s and a density e^{2u_0} on M , smooth, positive except maybe at a finite set of points which correspond to conical singularities of the metric $e^{2u_0}g$ on M such that

$$e^{2u_\epsilon} dv_g \rightharpoonup_\star e^{2u_0} dv_g \text{ on } \mathcal{M}(M(\rho)) \text{ as } \epsilon \rightarrow 0$$

with $M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$ and $\int_M e^{2u_0} dv_g = m_0$.

Proof. Let \tilde{N} be such that for $1 \leq i \leq N$,

$$1 \leq i \leq \tilde{N} \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

and

$$\tilde{N} + 1 \leq i \leq N \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \text{ is bounded.}$$

We now reintroduce the indices i we dropped in section 4.6 :

For $1 \leq i \leq \tilde{N}$, We let $\{q_{i,1}, \dots, q_{i,s_i}\} = \{\sigma^{-1}(p_{0,1}), \dots, \sigma^{-1}(p_{0,s_0})\}$ defined by Claim 26 and we recall that (4.52) holds in order to apply Proposition 3. We recall that

$$\Omega_i(\rho) = \mathbb{D}_{\frac{1}{\rho}} \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(\sigma(q_{i,j}))$$

and that ν_i is a measure without atoms which is a limit up to the extraction of a subsequence

$$e^{2\tilde{u}_\epsilon^i} dx \rightharpoonup_\star \nu_i \text{ in } \mathcal{M}(\Omega_i(\rho)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$.

For $\tilde{N} + 1 \leq i \leq N$, the notations just before Claim 31 define $e^{2\tilde{u}_i}$ as

$$e^{2\tilde{u}_\epsilon^i} \rightarrow e^{2\tilde{u}_i} \text{ in } \mathcal{C}^1(\mathbb{D}_{\frac{1}{\rho}}) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$.

We also take $\{p_1, \dots, p_s\}$ such that (4.31) holds and let

$$M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$$

and ν_0 be the measure without atoms such that

$$e^{2u_\epsilon} dv_g \rightharpoonup_\star \nu_0 \text{ in } \mathcal{M}(M(\rho)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$.

Then, we clearly have by (4.35) and (4.37) that

$$\int_{\mathbb{R}^2} d\nu_i \geq m_i \tag{4.88}$$

and by (4.36) and (4.38) that

$$\int_M dv_0 \geq m_0 . \quad (4.89)$$

Considering for $1 \leq i \leq N$ the set $M_i^\epsilon(\rho)$ such that

$$\left(H_{a_i^\epsilon, \alpha_i^\epsilon} \right)^{-1} \left(\widetilde{M_i^\epsilon(\rho)}^{l_i} \right) = \Omega_i(\rho) ,$$

(4.32), (4.37) give that

$$M(\rho) \cap M_i^\epsilon(\rho) = \emptyset \quad (4.90)$$

and (4.34) or (4.33) and (4.38) give that

$$i \neq j \Rightarrow M_i^\epsilon(\rho) \cap M_j^\epsilon(\rho) = \emptyset \quad (4.91)$$

for ϵ small enough.

By (4.90) and (4.91), we have for $\rho > 0$ and ϵ small enough

$$\int_M |\nabla \Phi_\epsilon|_g^2 dv_g \geq \mathbf{1}_{m_0 > 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g + \sum_{i=1}^N \int_{\Omega_i(\rho)} |\nabla \hat{\Phi}_\epsilon^i|^2 dx , \quad (4.92)$$

Then, applying (4.30) if $m_0 > 0$, (4.76) for $1 \leq i \leq \tilde{N}$, (4.79) and (4.81) for $\tilde{N} + 1 \leq i \leq N$, (4.88), (4.89) and the conservation of the mass (4.39)

$$\sum_{i=0}^N m_i = 1 ,$$

we get from (4.92) that

$$\begin{aligned} \Lambda_k(M, [g]) &= \lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_M |\nabla \Phi_\epsilon|_g^2 \\ &\geq \mathbf{1}_{m_0 > 0} \int_M \frac{|\nabla \Phi|_g^2}{|\Phi|} dv_g + \sum_{i=1}^{\tilde{N}} \int_{\mathbb{R}^2} \frac{|\nabla \hat{\Phi}_i|^2}{|\hat{\Phi}_i|} dx + \sum_{i=\tilde{N}+1}^N \int_{\mathbb{R}^2} |\nabla \hat{\Phi}_i|^2 dx \\ &\geq \mathbf{1}_{m_0 > 0} \left(\Lambda_k(M, [g]) \int_M dv_0 + \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g \right) \\ &\quad + \sum_{i=1}^{\tilde{N}} \left(\Lambda_k(M, [g]) \int_{\mathbb{R}^2} dv_i + \int_{\mathbb{R}^2} \frac{|\nabla |\hat{\Phi}_i||^2}{|\hat{\Phi}_i|} dx \right) \\ &\quad + \sum_{i=\tilde{N}+1}^N \Lambda_k(M, [g]) \int_{\mathbb{R}^2} e^{\hat{u}_i} \\ &\geq \Lambda_k(M, [g]) + \mathbf{1}_{m_0 > 0} \int_M \frac{|\nabla |\Phi||_g^2}{|\Phi|} dv_g + \sum_{i=1}^{\tilde{N}} \int_{\mathbb{R}^2} \frac{|\nabla |\hat{\Phi}_i||^2}{|\hat{\Phi}_i|} dx . \end{aligned}$$

Therefore, all the inequalities are equalities in Claim 23, Claim 30, Claim 31, (4.88) and (4.89) and we get for $1 \leq i \leq \tilde{N}$ that

$$|\hat{\Phi}_i|^2 = 1 \text{ on } \mathbb{R}^2$$

for $1 \leq i \leq \tilde{N}$, that

$$\int_{\mathbb{R}^2} dv_i = m_i$$

and if $m_0 > 0$, that

$$|\Phi|^2 = 1 \text{ on } M$$

and that

$$\int_M dv_0 = m_0.$$

Let $1 \leq i \leq \tilde{N}$. Then, $\hat{\Psi}_i = \hat{\Phi}_i$ and the equation (4.68) gives

$$0 = \frac{1}{2} \Delta \left(|\hat{\Phi}_i|^2 \right) = \Lambda_k(M, [g]) dv_i - |\nabla \hat{\Phi}_i|^2 dx$$

in a weak sense on $\mathbb{R}^2 \setminus \{q_{i,1}, \dots, q_{i,s_i}\}$. Then, $dv_i = \frac{|\nabla \hat{\Phi}_i|^2}{\Lambda_k(M, [g])} dx$ that is v_i is absolutely continuous with respect to dx and

$$\Delta \hat{\Phi}_i = |\nabla \hat{\Phi}_i|^2 \hat{\Phi}_i$$

which means that $\hat{\Phi}_i$ is weakly harmonic on $\mathbb{R}^2 \setminus \{q_{i,1}, \dots, q_{i,s_i}\}$. Coming back to the sphere with the pullback by σ , by Sacks-Uhlenbeck (see [100]), since $\int_{S^2} |\nabla \check{\Phi}_i|^2 dv_h < +\infty$, we can extend $\check{\Phi}_i$ as a weakly harmonic map on S^2 . By the regularity theory for weakly harmonic maps of Hélein [51], $\check{\Phi}_i$ is smooth and harmonic on S^2 . Setting $e^{2\check{u}_i} = \frac{|\nabla \check{\Phi}_i|^2}{\Lambda_k(M, [g])}$ gives the first part of the claim for $1 \leq i \leq \tilde{N}$.

For $\tilde{N} + 1 \leq i \leq N$, the convergence (4.78) ends the proof of the first part of the claim.

If $m_0 > 0$, then, $\Psi = \Phi$ and the equation (4.25) gives

$$0 = \frac{1}{2} \Delta_g \left(|\Phi|^2 \right) = \Lambda_k(M, [g]) dv_0 - |\nabla \Phi|_g^2 dv_g$$

in a weak sense on $M \setminus \{p_1, \dots, p_s\}$. Then, $dv_0 = \frac{|\nabla \Phi|_g^2}{\Lambda_k(M, [g])} dx$ that is v_0 is absolutely continuous with respect to dv_g and

$$\Delta_g \Phi = |\nabla \Phi|_g^2 \Phi$$

which means that Φ is weakly harmonic on $M \setminus \{p_1, \dots, p_s\}$. By Sacks-Uhlenbeck (see [100]), since $\int_M |\nabla \Phi|_g^2 dv_g < +\infty$, we can extend Φ as a weakly harmonic map on M . By the regularity theory for weakly harmonic maps of Hélein [51], Φ is smooth and harmonic on M . Setting $e^{2u_0} = \frac{|\nabla \Phi|_g^2}{\Lambda_k(M, [g])}$ gives the second part of the claim. ◇

4.7.2 Gaps and no concentration

We prove now by contradiction that $N = 0$, so that the maximizing sequence $\{e^{2u_\epsilon} dv_g\}$ does not have any concentration points. Therefore, by Proposition 4 with $m_0 = 1$, the proof of Theorem 9 will follow.

We now assume that $N \geq 1$ and we use Proposition 4 and the gap assumption (strictness of (4.3)) in order to get a contradiction.

For $1 \leq i \leq N$, let θ_i be the maximal integer such that

$$\frac{\Lambda_{\theta_i}(\mathbb{S}^2)}{m_i} < \Lambda_k(M, [g]) \quad (4.93)$$

and let θ_0 be the maximal integer such that

$$\frac{\Lambda_{\theta_0}(M, [g])}{m_0} < \Lambda_k(M, [g]) \quad (4.94)$$

if $m_0 > 0$. We set $\theta_0 = -1$ if $m_0 = 0$. We get that for $i \in \{1, \dots, N\}$,

$$\Lambda_{\theta_i+1}(\mathbb{S}^2) \geq m_i \Lambda_k(M, [g]) \quad (4.95)$$

and

$$\Lambda_{\theta_0+1}(M, [g]) \geq m_0 \Lambda_k(M, [g]) \quad (4.96)$$

Then, by the spectral gap assumption of the theorem, we have that

$$\sum_{i=0}^N (\theta_i + 1) \geq k + 1. \quad (4.97)$$

Indeed, if $\sum_{i=0}^N (\theta_i + 1) \leq k$, the spectral gap gives that

$$\sum_{i=1}^N \Lambda_{\theta_i+1}(\mathbb{S}^2) + \Lambda_{\theta_0+1}(M, [g]) < \Lambda_k(M, [g])$$

and this contradicts (4.39), (4.95) and (4.96).

Now, we will define at least $k + 1$ test functions for the min-max characterization (4.2.1) of $\lambda_\epsilon = \lambda_k(M, e^{2u_\epsilon} g)$.

Let $1 \leq i \leq N$. We denote by $(\varphi_i^0, \dots, \varphi_i^{\theta_i})$ an orthonormal family in $L^2(M, e^{2u_0} dv_g)$ if $i = 0$ and in $L^2(\mathbb{S}^2, e^{2u_i} dv_h)$ if $i \neq 0$, such that if $0 \leq j \leq \theta_i$, φ_i^j is an eigenfunction for $\lambda_j(M, e^{2u_0} g)$ if $i = 0$ and for $\lambda_j(\mathbb{S}^2, e^{2u_i} h)$ if $i \neq 0$. Such functions exist by Proposition 4 and lie in \mathcal{C}^1 .

We fix $\rho > 0$. We denote by η_i some function defined with Claim 18 by

- $\eta_0 \in \mathcal{C}_c^\infty(M(\sqrt{\rho}))$, $\eta_0 \geq 1$ on $M(\rho)$ and $\int_M |\nabla \eta_0|_g^2 dv_g \leq \frac{C}{\ln(\frac{1}{\rho})}$.
- If $i \neq 0$, $\eta_i \in \mathcal{C}_c^\infty(S_i(\sqrt{\rho}))$, $\eta_i \geq 1$ on $S_i(\rho)$ and $\int_{\mathbb{S}^2} |\nabla \eta_i|_h^2 dv_h \leq \frac{C}{\ln(\frac{1}{\rho})}$.

We set for $0 \leq i \leq N$ and $0 \leq j \leq \theta_i$ some test functions $\tilde{\zeta}_i^j$, defined by

$$\tilde{\zeta}_0^j = \eta_0 \varphi_0^j \text{ on } M$$

and if $i \neq 0$, $\tilde{\zeta}_i^j$ depends on ϵ and satisfies for any $\epsilon > 0$

$$\left(\tilde{\zeta}_i^j \right)_\epsilon^i = \eta_i \varphi_i^j \text{ on } \mathbb{S}^2$$

extended by 0 on M .

Note that all the test functions $\tilde{\zeta}_i^j$ lie in \mathcal{C}^1 and are uniformly bounded. Note also that by (4.90) and (4.91), if ϵ small enough,

$$i \neq i' \Rightarrow \text{supp}(\tilde{\zeta}_i^j) \cap \text{supp}(\tilde{\zeta}_{i'}^{j'}) = \emptyset$$

for $i, i' \in \{0, \dots, N\}$, $0 \leq j \leq \theta_i$ and $0 \leq j' \leq \theta_{i'}$. For $1 \leq i \leq N$, we let E_i the vector space spanned by $(\xi_i^0, \xi_i^1, \dots, \xi_i^{\theta_i})$ and with (4.97) we deduce that

$$\lambda_\epsilon \leq \max_{0 \leq i \leq N} \sup_{\xi \in E_i \setminus \{0\}} \frac{\int_M |\nabla \xi|_g^2 dv_g}{\int_M \xi^2 e^{2u_\epsilon} dv_g}. \quad (4.98)$$

Let $i \in \{1, \dots, N\}$. For $\xi = \sum_{j=0}^{\theta_i} \mu_j \xi_i^j \in E_i$, with $\mu_j \in \mathbb{R}$ and $\sum_j \mu_j^2 = 1$, we get

$$\int_M |\nabla \xi|_g^2 dv_g = \int_{\mathbb{S}^2} \left| \nabla \left(\eta_i \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j \right) \right|_h^2 dv_h$$

and denoting $\varphi = \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j$, we have

$$\begin{aligned} \int_M |\nabla \xi|_g^2 dv_g &= \int_{\mathbb{S}^2} (\eta_i)^2 |\nabla \varphi|_h^2 dv_h + 2 \int_{\mathbb{S}^2} \eta_i \varphi \langle \nabla \eta_i, \nabla \varphi \rangle dv_h + \int_{\mathbb{S}^2} \varphi^2 |\nabla \eta_i|_h^2 dv_h \\ &\leq \int_{\mathbb{S}^2} |\nabla \varphi|_h^2 dv_h + 2 \|\eta_i \varphi\|_\infty \left(\int_{\mathbb{S}^2} |\nabla \varphi|_h^2 dv_h \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^2} |\nabla \eta_i|_h^2 dv_h \right)^{\frac{1}{2}} \\ &\quad + \|\varphi\|_\infty^2 \int_{\mathbb{S}^2} |\nabla \eta_i|_h^2 dv_h \\ &\leq \int_{\mathbb{S}^2} |\nabla \varphi|_h^2 dv_h + O \left(\frac{1}{\sqrt{\ln(\frac{1}{\rho})}} \right) \text{ as } \rho \rightarrow 0. \end{aligned}$$

We also have that

$$\int_M \xi^2 e^{2u_\epsilon} dv_g = \int_{\mathbb{S}^2} \eta_i^2 \varphi^2 e^{2u_\epsilon} dv_h.$$

By Proposition 4, we get that

$$\int_M \xi^2 e^{2u_\epsilon} dv_g = \int_{\mathbb{S}^2} \eta_i^2 \varphi^2 e^{2u_\epsilon} dv_h + o(1) \text{ as } \epsilon \rightarrow 0$$

so that

$$\lim_{\epsilon \rightarrow 0} \int_M \xi^2 e^{2u_\epsilon} dv_g \geq \int_{\mathbb{S}^2} \varphi^2 e^{2u_\epsilon} dv_h + o(1) \text{ as } \rho \rightarrow 0.$$

The same work can be done on for $\xi \in E_0$, so that passing to the limit as $\epsilon \rightarrow 0$ and then as $\rho \rightarrow 0$ in (4.98), we get

$$\Lambda_k(M, [g]) \leq \max \left\{ \max_{1 \leq i \leq N} \sup_{\varphi \in F_i \setminus \{0\}} \frac{\int_{\mathbb{S}^2} |\nabla \varphi|_h^2 dv_h}{\int_{\mathbb{S}^2} \varphi^2 e^{2u_i} dv_h}, \sup_{\varphi \in F_0 \setminus \{0\}} \frac{\int_M |\nabla \varphi|_g^2 dv_g}{\int_M \varphi^2 e^{2u_0} dv_g} \right\}$$

where F_i is the space spanned by $\varphi_i^0, \dots, \varphi_i^{\theta_i}$. Therefore,

$$\begin{aligned} \lambda_k(M, [g]) &\leq \max \left\{ \max_{1 \leq i \leq N} \lambda_{\theta_i}(\mathbb{S}^2, e^{2u_i} h), \lambda_{\theta_0}(M, e^{2u_0} g) \right\} \\ &\leq \max \left\{ \max_{1 \leq i \leq N} \frac{\Lambda_{\theta_i}(\mathbb{S}^2)}{m_i}, \frac{\Lambda_{\theta_0}(M, [g])}{m_0} \right\} \end{aligned}$$

which contradicts (4.95) and (4.96). Therefore, there is no concentration of $\{e^{2u_\epsilon} dv_g\}$.

Therefore, $N = 0$ and by Proposition 4 with $m_0 = 1$, Theorem 9 follows.

4.8 Maximal metrics for the topological invariant

We prove Theorem 8 in this section. Notice that light modifications of the proof allow us to prove that if we have that (4.3) is strict, the set of maximal metrics for $\Lambda_k(M, [g])$ is compact, and if we have that (4.1) is strict, the set of maximal metrics for $\Lambda_k(\gamma)$ is compact.

Let $\gamma \geq 1$ and $[g_\alpha]$ be a sequence of conformal classes on a compact oriented manifold without boundary of genus γ such that

$$\lambda_\alpha = \Lambda_k(M, [g_\alpha]) \rightarrow \Lambda_k(\gamma) \text{ as } \alpha \rightarrow +\infty, \quad (4.99)$$

where g_α denotes the unique metric with constant curvature 0 for $\gamma = 1$ or -1 for $\gamma \geq 2$ in its conformal class. By the gap assumption of Theorem 8, we have in particular that

$$\Lambda_k(M, [g_\alpha]) > \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} \Lambda_{k-j}(M, [g_\alpha]) + \sum_{m=1}^s \Lambda_{i_m}(\mathbb{S}^2, [can])$$

for α large enough. This gives by Theorem 9 some unit volume metrics \tilde{g}_α and smooth harmonic maps $\phi_\alpha : (M, g_\alpha) \rightarrow \mathbb{S}^{n_\alpha}$ for some $n_\alpha > 0$ such that

$$\tilde{g}_\alpha = \frac{|\nabla \Phi_\alpha|_{g_\alpha}^2}{\lambda_\alpha} g_\alpha$$

and $\lambda_k(M, \tilde{g}_\alpha) = \Lambda_k(M, [g_\alpha])$. Since the multiplicity of λ_k is bounded by a constant which only depends on k and γ , we can assume that $n = n_\alpha$ is fixed.

Let us mention the following result by [88] and [100] often used in the proof of Theorem 8.

Proposition 5. *Let (M, g) be a compact Riemannian surface without boundary. We refer to the notations introduced in Section 4.2.1 for the metric g . Let $q_1, \dots, q_t \in M$. Let $\Phi_\alpha : (M_\alpha, g_\alpha) \rightarrow \mathbb{S}^n$ be a sequence of harmonic maps on open sets $M_\alpha \subset M$ such that*

- *For any $\rho > 0$, there exists $\alpha_\rho > 0$ such that for any $\alpha > \alpha_\rho$, $M_\alpha \supset M \setminus \bigcup_{i=1}^t B_g(q_i, \rho)$.*
- *For any $\rho > 0$, $g_\alpha \rightarrow g$ in $M \setminus \bigcup_{i=1}^t B_g(q_i, \rho)$ as $\alpha \rightarrow +\infty$.*
- *$\limsup_{\alpha \rightarrow +\infty} \int_{M_\alpha} |\nabla \Phi_\alpha|_{g_\alpha}^2 dv_{g_\alpha} < +\infty$.*

Then, up to the extraction of a subsequence there exist

- *Some harmonic map $\Phi : M \rightarrow \mathbb{S}^n$*
- *Sequences of points $p_\alpha^1, \dots, p_\alpha^s$ of M converging to some points p^1, \dots, p^s of $M \setminus \{q_1, \dots, q_t\}$ as $\alpha \rightarrow +\infty$ and sequences of scales $\delta_\alpha^1, \dots, \delta_\alpha^s$ converging to 0 as $\alpha \rightarrow +\infty$ with*

$$\frac{d_g(p_\alpha^i, p_\alpha^j)}{\delta_\alpha^i + \delta_\alpha^j} + \frac{\delta_\alpha^i}{\delta_\alpha^j} + \frac{\delta_\alpha^j}{\delta_\alpha^i} \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty \quad (4.100)$$

- *Some non constant harmonic maps $\omega_1, \dots, \omega_s : \mathbb{S}^2 \rightarrow \mathbb{S}^n$*

such that

$$\int_M |\nabla \Phi|_g^2 dv_g + \sum_{i=1}^s \int_{\mathbb{S}^2} |\nabla \omega_i|_h^2 dv_h = \mathcal{E} \quad (4.101)$$

where

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus \bigcup_{i=1}^t B_g(q_i, \rho)} |\nabla \Phi_\alpha|_{g_\alpha}^2 dv_{g_\alpha}$$

and for any $\rho > 0$,

$$|\nabla \Phi_\alpha|_{g_\alpha}^2 dv_{g_\alpha} - |\nabla \Phi|_g^2 dv_g \rightharpoonup_\star 0 \text{ on } M(\rho), \quad (4.102)$$

$$|\nabla \hat{\Phi}_\alpha^i|^2 dx - |\nabla \hat{\omega}_i|^2 dx \rightharpoonup_\star 0 \text{ on } \Omega_i(\rho), \quad (4.103)$$

where we denote the sets

$$M(\rho) = M \setminus \left(\bigcup_{i=1}^t B_g(q_i, \rho) \cup \bigcup_{z \in Z(M \setminus \bigcup_{i=1}^t B_g(q_i, \rho), |\nabla \Phi_\alpha|_{g_\alpha}^2 dv_{g_\alpha})} B_g(z, \rho) \right)$$

$$\Omega_i(\rho) = \mathbb{D}_{\frac{1}{\rho}} \setminus \bigcup_{z \in Z(\mathbb{D}_{\frac{1}{\rho}}, |\nabla \hat{\Phi}_\alpha^i|^2 dx)} \mathbb{D}_\rho(z)$$

and the functions on $\Omega_i(\rho) \subset \mathbb{R}^2$

$$\hat{\Phi}_\alpha^i(x) = \widetilde{\Phi}_\alpha^{l_i}(\delta_\alpha^i x + \tilde{p}_\alpha^{l_i}) \text{ and } \hat{\omega}_i = \omega_i \circ \sigma^{-1},$$

where $1 \leq l_i \leq L$ is chosen such that $p^i \in \omega_{l_i}$ and σ is the stereographic projection with respect to some pole $p \in \mathbb{S}^2$.

We first assume that $g_\alpha \rightarrow g$ as $\alpha \rightarrow +\infty$ for some metric g with constant curvature 0 for $\gamma = 1$ and -1 for $\gamma \geq 2$. We apply Proposition 5 for $M_\alpha = M$, Φ_α , g_α and g . Notice that the use of Proposition 5 together with the assumption that (4.1) is strict follows exactly the same paths as the use of Proposition 4 together with the assumption that (4.3) is strict in order to prove that the maximizing sequences do not have any concentration points. Therefore, one can easily contradict the assumption that (4.1) is strict in this case.

We assume now that the sequence of conformal classes $[g_\alpha]$ degenerates in the following sense :

- If $\gamma = 1$, in the case of the torus, this means that $b_\alpha \rightarrow +\infty$ if (a_α, b_α) denotes the real parameters $0 \leq a_\alpha \leq \frac{1}{2}$, $b_\alpha > 0$, $a_\alpha^2 + b_\alpha^2 = 1$ such that (M, g_α) is isometric to the flat torus $\mathbb{R}^2/\Gamma_\alpha$ where Γ_α denotes the lattice generated by $(0, 1)$ and (a_α, b_α)
- If $\gamma \geq 2$, in the hyperbolic case, this means that the injectivity radius $i_{g_\alpha}(M) \rightarrow 0$ as $\alpha \rightarrow +\infty$ so that there exists closed geodesics whose length goes to 0.

Let's tackle both cases in order to contradict the assumption that (4.1) is strict.

4.8.1 Case of the torus

For $\gamma = 1$, we identify M and $T_\alpha = \mathbb{R}^2/\Gamma_\alpha$ and we let for $0 \leq r \leq s \leq b_\alpha$

$$T_\alpha(r, s) = \{(x, y) \in T_\alpha; r \leq y \leq s\}.$$

For sequences $\{r_\alpha\}$ and $\{s_\alpha\}$, $r_\alpha \ll s_\alpha$ means $s_\alpha - r_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. Then, we claim that

Claim 32. *If some sequences $\{r_\alpha^i\}$ and $\{s_\alpha^i\}$ for $1 \leq i \leq t$ satisfy*

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \dots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

and

$$m_j = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\hat{g}_\alpha}(T_\alpha(r_\alpha^i, s_\alpha^i)) > 0$$

for $1 \leq i \leq t$, then $t \leq k$.

Proof

We proceed by contradiction and assume that we have such sequences with $t \geq k+1$. Let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(r_\alpha^{i+1} - s_\alpha^i)$ as $\alpha \rightarrow +\infty$ for $0 \leq i \leq t$. We set for $1 \leq i \leq t$

$$\eta_\alpha^i = \begin{cases} 1 & r_\alpha^i \leq y \leq s_\alpha^i \\ \frac{y - r_\alpha^i + \theta_\alpha}{\theta_\alpha} & r_\alpha^i - \theta_\alpha \leq y \leq r_\alpha^i \\ \frac{s_\alpha^i + \theta_\alpha - y}{\theta_\alpha} & s_\alpha^i \leq y \leq s_\alpha^i + \theta_\alpha \\ 0 & y \geq s_\alpha^i + \theta_\alpha \text{ or } y \leq r_\alpha^i - \theta_\alpha \end{cases}$$

Then,

$$\int_{T_\alpha} |\nabla \eta_\alpha^i|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha} = \int_{T_\alpha} |\nabla \eta_\alpha^i|^2 dx = \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \rightarrow +\infty,$$

$$\int_{T_\alpha} (\eta_\alpha^i)^2 dv_{\tilde{g}_\alpha} \geq m_j + o(1) \text{ as } \alpha \rightarrow +\infty.$$

Taking these at least $k+1$ functions with pairwise disjoint support for the variational characterization (4.2.1) of $\lambda_\alpha = \lambda_k(M, \tilde{g}_\alpha)$ leads to

$$\lambda_\alpha \leq \max_{1 \leq i \leq k+1} \frac{\int_{T_\alpha} |\nabla \eta_\alpha^i|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_{T_\alpha} (\eta_\alpha^i)^2 dv_{\tilde{g}_\alpha}} = o(1) \text{ as } \alpha \rightarrow +\infty$$

which contradicts (4.99).

◇

Now, we prove that up to a vertical translation on T_α , there exists sequences $0 \ll r_\alpha \ll s_\alpha \ll b_\alpha$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(r_\alpha, s_\alpha)) = 1. \quad (4.104)$$

Indeed, denying (4.104) would mean that for any sequence $1 \ll u_\alpha \ll v_\alpha \ll b_\alpha$,

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(u_\alpha, v_\alpha)) > 0.$$

Taking for $1 \leq j \leq k+1$ $y_\alpha^j = \frac{j}{k+2}b_\alpha$ and $\theta_\alpha = \sqrt{b_\alpha}$ gives for $1 \leq j \leq k+1$

$$m_j = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(y_\alpha^j - \theta_\alpha, y_\alpha^j + \theta_\alpha)) > 0$$

so that the $k+1$ test functions for $\lambda_\alpha = \lambda_k(M, \tilde{g}_\alpha)$ with pairwise disjoint support

$$\eta_\alpha^j = \begin{cases} 1 & y_\alpha^j - \theta_\alpha \leq y \leq y_\alpha^j + \theta_\alpha \\ \frac{y - y_\alpha^j + 2\theta_\alpha}{\theta_\alpha} & y_\alpha^j - 2\theta_\alpha \leq y \leq y_\alpha^j - \theta_\alpha \\ \frac{y_\alpha^j + 2\theta_\alpha - y}{\theta_\alpha} & y_\alpha^j + \theta_\alpha \leq y \leq y_\alpha^j + 2\theta_\alpha \\ 0 & y \geq y_\alpha^j + 2\theta_\alpha \text{ or } y \leq y_\alpha^j - 2\theta_\alpha \end{cases}$$

would satisfy

$$\begin{aligned} \int_{T_\alpha} |\nabla \eta_j^\alpha|^2 dv_{\tilde{g}_\alpha} &= \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \rightarrow +\infty, \\ \int_{T_\alpha} (\eta_j^\alpha)^2 dv_{\tilde{g}_\alpha} &\geq m_j + o(1) \text{ as } \alpha \rightarrow +\infty, \end{aligned}$$

so that $\lambda_\alpha = o(1)$. This contradicts (4.99).

We take a vertical translation of T_α so that (4.104) holds. Then, by Claim 32, we can take t the maximal integer such that there exists sequences

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \dots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

with

$$m_j = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(r_\alpha^j, s_\alpha^j)) > 0$$

and

$$\sum_{j=1}^t m_j = 1.$$

We define a sequence $r_\alpha^j < y_\alpha^j < s_\alpha^j$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(r_\alpha^j, y_\alpha^j)) = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(y_\alpha^j, s_\alpha^j)) = \frac{m_j}{2}$$

and

$$\Psi_\alpha^j(x, y) = \frac{1}{1 + e^{2(y-y_\alpha^j)}} (e^{2(y-y_\alpha^j)} \cos(2\pi x), e^{2(y-y_\alpha^j)} \sin(2\pi x), e^{2(y-y_\alpha^j)} - 1)$$

for $0 \leq x \leq 1$ and $0 \leq y \leq b_\alpha$. We consider the harmonic map $\check{\Phi}_\alpha^j = \Phi_\alpha \circ (\Psi_\alpha^j)^{-1}$ on \mathbb{S}^2 . We let $\theta_\alpha \rightarrow +\infty$ such that $\theta_\alpha = o(r_\alpha^{j+1} - s_\alpha^j)$ for all $0 \leq j \leq t$. Then,

$$S_\alpha^j = \Psi_\alpha^j(T_\alpha(r_\alpha^j - \theta_\alpha, r_\alpha^j + \theta_\alpha))$$

exhausts \mathbb{S}^2 and

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\check{g}_\alpha}(S_\alpha^j) = m_j,$$

where $\check{g}_\alpha = (\Psi_\alpha^j)_* \tilde{g}_\alpha$.

Now, we can apply Proposition 5 for the manifold (\mathbb{S}^2, h) and the sequence $\check{\Phi}_\alpha^j : (S_\alpha^j, \check{g}_\alpha) \rightarrow \mathbb{S}^n$ of harmonic maps. In order to define suitable test functions which naturally extend to the surface we have to prove that $\mathbf{1}_{S_\alpha^j} |\nabla \check{\Phi}_\alpha^j|_h^2 dv_h$ does not concentrate in the poles $(0, 0, 1)$ and $(0, 0, -1)$. Let's prove it by contradiction : if for instance we have

$$\mathbf{1}_{S_\alpha^j} |\nabla \check{\Phi}_\alpha^j|_h^2 dv_h \rightharpoonup_* m\delta_{(0,0,1)} + \nu \text{ on } \mathbb{S}^2$$

with $m > 0$, and $\nu(\{(0, 0, 1)\}) = 0$, then, $\int_{\mathbb{S}^2} d\nu > 0$ and up to the extraction of a subsequence, we can build $c_\alpha^j \ll y_\alpha^j$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(r_\alpha^j - \theta_\alpha, c_\alpha^j)) = m,$$

so that if we set $\overline{r}_\alpha = y_\alpha^j + \tau_\alpha$ and $\overline{s}_\alpha = c_\alpha^j + \tau_\alpha$ with $\tau_\alpha = \sqrt{y_\alpha^j - c_\alpha^j}$, we have

$$m_j^1 = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(r_\alpha^j - \theta_\alpha, \overline{s}_\alpha)) > 0$$

$$m_j^2 = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(T_\alpha(\overline{r}_\alpha, s_\alpha^j + \theta_\alpha)) > 0$$

with $m_j^1 + m_j^2 = m_j$ and this contradicts the maximality of t .

Therefore, following Proposition 5 and the computations of Section 4.7.2, some suitable test functions associated to $\check{\Phi}_\alpha^j$ with $1 \leq j \leq t$ will be well defined for the variational characterization (4.2.1) of $\lambda_\alpha = \lambda_k(M, \tilde{g}_\alpha)$. By the strictness of (4.1) which reads on the torus as

$$\Lambda_k(\gamma) > \max_{i_1 + \dots + i_s = k} \sum_{m=1}^s \Lambda_{i_m}(0),$$

we can define at least $k + 1$ test functions which would give a contradiction.

4.8.2 The hyperbolic case

Now, we assume that $\gamma \geq 2$. We let $\gamma_\alpha^1, \dots, \gamma_\alpha^s$ the closed geodesics whose length $l_\alpha^1, \dots, l_\alpha^s$ go to 0 as $\alpha \rightarrow +\infty$, where $1 \leq s \leq 3\gamma - 3$ ([55], IV, lemma 4.1). The collar lemma ([115], lemma 4.2) gives for $1 \leq i \leq s$ an open neighbourhood P_α^i of γ_α^i isometric to the cylinder

$$\{(t, \theta), -\mu_\alpha^i < t < \mu_\alpha^i, 0 \leq \theta < 2\pi\}$$

endowed with the metric

$$\left(\frac{l_\alpha^i}{2\pi \cos\left(\frac{l_\alpha^i t}{2\pi}\right)} \right)^2 (dt^2 + d\theta^2)$$

with

$$\mu_\alpha^i = \frac{\pi}{l_\alpha^i} \left(\pi - 2 \arctan \left(\sinh \left(\frac{l_\alpha^i}{2} \right) \right) \right)$$

and with identification of the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$. Notice that the geodesic γ_α^i corresponds to the line $\{t = 0\}$. Note that in the following, we identify P_α^i with the cylinder.

We denote $M_\alpha^1, \dots, M_\alpha^r$ the connected components of $M \setminus \bigcup_{i=1}^s P_\alpha^i$ so that

$$M = \left(\bigcup_{i=1}^s P_\alpha^i \right) \cup \left(\bigcup_{j=1}^r M_\alpha^j \right)$$

is a disjoint union. For $-\mu_\alpha^i < a < b < \mu_\alpha^i$, we denote

$$P_\alpha^i(a, b) = \{(t, \theta); a < t < b\}$$

and for $c = \{c^{i,-}, c^{i,+}\}_{1 \leq i \leq s}$, we denote $M_\alpha^j(c)$ the connected component of $M \setminus \bigcup_{i=1}^s P_\alpha^i$ $P_\alpha^i(-\mu_\alpha^i + c^{i,-}, \mu_\alpha^i - c^{i,+})$ which contains M_α^j . We also denote $a_\alpha \ll b_\alpha$ if two sequences a_α and b_α satisfy $b_\alpha - a_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$.

Claim 33. If for integers $t_i \geq 0$, some sequences $a_\alpha^{i,l}, b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \dots, r\}$ satisfy

$$\begin{aligned} -\mu_\alpha^i &\ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \dots \\ &\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_i+1} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i \end{aligned}$$

and for $1 \leq i \leq s, 1 \leq l \leq t_i, j \in J$,

$$m_{i,l} = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha(a_\alpha^{i,l}, b_\alpha^{i,l})) > 0$$

$$m_j = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(M_\alpha^j(c_\alpha)) > 0,$$

then, $\sum_{i=1}^s t_i + |J| \leq k$.

Proof

By contradiction, we assume that there exist such sequences with $\sum_{i=1}^s t_i + |J| \geq k + 1$. Let $\theta_\alpha \rightarrow +\infty$ such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $1 \leq i \leq s$ and $0 \leq l \leq t_i$. We let $\eta_\alpha^{i,l}$ be such that $\text{supp}(\eta_\alpha^{i,l}) \subset P_\alpha^i$ and

$$\eta_\alpha^{i,l} = \begin{cases} 1 & a_\alpha^{i,l} \leq t \leq b_\alpha^{i,l} + \theta_\alpha \\ \frac{t - a_\alpha^{i,l} + \theta_\alpha}{\theta_\alpha} & a_\alpha^{i,l} - \theta_\alpha \leq t \leq a_\alpha^{i,l} \\ \frac{b_\alpha^{i,l} + \theta_\alpha - t}{\theta_\alpha} & b_\alpha^{i,l} \leq t \leq b_\alpha^{i,l} + \theta_\alpha \\ 0 & t \geq b_\alpha^{i,l} + \theta_\alpha \text{ or } t \leq a_\alpha^{i,l} - \theta_\alpha \end{cases}$$

and η_α^j such that $\text{supp}(\eta_\alpha^j) \subset M_\alpha^j(c_\alpha + \theta_\alpha)$ and if $\{t = \mu_\alpha^i\}$ is on the boundary of M_α^j ,

$$\eta_\alpha^j = \begin{cases} 1 & \mu_\alpha^i - c_\alpha^{i,+} \leq t \leq \mu_\alpha^i \\ \frac{t - \mu_\alpha^i + c_\alpha^{i,+} + \theta_\alpha}{\theta_\alpha} & \mu_\alpha^i - c_\alpha^{i,+} - \theta_\alpha \leq t \leq \mu_\alpha^i - c_\alpha^{i,+} \end{cases}$$

and we proceed the same way for the symmetric case $\{t = -\mu_\alpha^i\}$ with $c_\alpha^{i,-}$. Taking these at least $k + 1$ test functions with pairwise disjoint support for the variational characterization (4.2.1) of $\lambda_\alpha = \lambda_k(M, \tilde{g}_\alpha)$, we get

$$\lambda_\alpha \leq \max \left(\max_{\substack{1 \leq i \leq s \\ 1 \leq l \leq t_i}} \frac{\int_M |\nabla \eta_\alpha^{i,l}|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_M (\eta_\alpha^{i,l})^2 dv_{\tilde{g}_\alpha}}, \max_{j \in J} \frac{\int_M |\nabla \eta_\alpha^j|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_M (\eta_\alpha^j)^2 dv_{\tilde{g}_\alpha}} \right).$$

Then $\lambda_\alpha \leq o(1)$ which contradicts (4.99).

◇

We now prove that the set of such sequences such that

$$\sum_{i=1}^s \sum_{l=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1$$

is not empty.

Claim 34. We let I_0 be the set of indices $i \in \{1, \dots, s\}$ such that there exists a sequence $0 \ll c_\alpha^i \ll \mu_\alpha^i$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(-\mu_\alpha^i + c_\alpha^i, \mu_\alpha^i - c_\alpha^i)) = 0$$

and $I_1 = \{1, \dots, s\} \setminus I_0$. Then, there exist sequences $c_\alpha^{i,\pm} \rightarrow +\infty$ $0 \ll c_\alpha^{i,\pm} \ll \mu_\alpha^i$ for $1 \leq i \leq s$ and sequences a_α^i, b_α^i for $i \in I_1$ with

$$-\mu_\alpha^i + c_\alpha^{i,+} \ll a_\alpha^i \ll b_\alpha^i \ll \mu_\alpha^i - c_\alpha^{i,-},$$

such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(-\mu_\alpha^i + c_\alpha^{i,-}, \mu_\alpha^i - c_\alpha^{i,+})) = 0$$

for $i \in I_0$,

$$\lim_{\alpha \rightarrow +\infty} \sum_{i=1}^s \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(a_\alpha^i, b_\alpha^i)) > 0$$

for $i \in I_1$ and

$$\lim_{\alpha \rightarrow +\infty} \sum_{i \in I_1} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(a_\alpha^i, b_\alpha^i)) + \sum_{j=1}^r \text{Vol}_{\tilde{g}_\alpha}(M_\alpha^j(c_\alpha)) = 1.$$

Proof

We proceed by contradiction, assuming the opposite holds. Then $I_1 \neq \emptyset$ and we set for $i \in I_1$ and $1 \leq j \leq k+1$

$$\mu_\alpha^i - c_\alpha^{i,+} = \mu_\alpha^i - c_\alpha^{i,-} = t_\alpha^{i,j} + \theta_\alpha$$

$$b_\alpha^j = -a_\alpha^j = t_\alpha^j - \theta_\alpha$$

where $t_\alpha^j = \frac{j\mu_\alpha^j}{k+2}$ and $\theta_\alpha \rightarrow +\infty$ satisfies $\theta_\alpha = o(\mu_\alpha^i)$. Then, by assumption,

$$\sum_{i=1}^s \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha} \left(P_\alpha^i(-t_\alpha^{i,j} - \theta_\alpha, -t_\alpha^{i,j} + \theta_\alpha) \cup P_\alpha^i(t_\alpha^{i,j} - \theta_\alpha, t_\alpha^{i,j} + \theta_\alpha) \right) > 0$$

for any $1 \leq j \leq k+1$. We now set η_α^j some test functions for the variational characterization (4.2.1) of $\lambda_\alpha = \lambda_k(M, \tilde{g}_\alpha)$ with pairwise disjoint support defined such that $\text{supp}(\eta_\alpha^j) \subset \bigcup_{i \in I_1} P_\alpha^i$,

η_α^i is an even function on P_α^i and

$$\eta_\alpha^{i,j} = \begin{cases} 0 & 0 \leq t \leq t_\alpha^{i,j} - 2\theta_\alpha \\ \frac{t - t_\alpha^{i,j} + 2\theta_\alpha}{\theta_\alpha} & t_\alpha^{i,j} - 2\theta_\alpha \leq t \leq t_\alpha^{i,j} - \theta_\alpha \\ 1 & t_\alpha^{i,j} - \theta_\alpha \leq t \leq t_\alpha^{i,j} + \theta_\alpha \\ \frac{t_\alpha^{i,j} + 2\theta_\alpha - t}{\theta_\alpha} & t_\alpha^{i,j} + \theta_\alpha \leq t \leq t_\alpha^{i,j} + 2\theta_\alpha \\ 0 & t_\alpha^{i,j} + 2\theta_\alpha \leq t \leq \mu_\alpha^i \end{cases}$$

With these $k+1$ test functions, we easily prove that $\lambda_\alpha \leq o(1)$, which contradicts (4.99). \diamond

Thanks to Claim 33 and Claim 34 there exist for $1 \leq i \leq s$ some integers $t_i \geq 0$ sequences $a_\alpha^{i,l}, b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \dots, r\}$ satisfying $c_\alpha^{i,\pm} < \mu_\alpha^i$,

$$\begin{aligned} -\mu_\alpha^i &\ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \dots \\ &\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_i+1} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i \end{aligned}$$

and for $1 \leq i \leq s, 1 \leq l \leq t_i, j \in J$,

$$m_{i,l} = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha(a_\alpha^{i,l}, b_\alpha^{i,l})) > 0$$

$$m_j = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(M_\alpha^j(c_\alpha)) > 0,$$

with

$$\sum_{i=1}^s \sum_{m=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1$$

such that $\sum_{i=1}^s t_i$ is maximal.

For fixed $1 \leq i \leq s$ and $1 \leq l \leq t_i$, we focus on the asymptotic behaviour of Φ_α on the cylinder $P_\alpha^i(a_\alpha^{i,l}, b_\alpha^{i,l})$. We define a sequence $t_\alpha^{i,l}$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha(a_\alpha^{i,l}, t_\alpha^{i,l})) = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha(t_\alpha^{i,l}, b_\alpha^{i,l})) = \frac{m_{i,l}}{2}.$$

We set

$$\Psi_\alpha^{i,l}(t, \theta) = \frac{1}{1 + e^{2(t-t_\alpha^{i,l})}} (e^{2(t-t_\alpha^{i,l})} \cos(\theta), e^{2(t-t_\alpha^{i,l})} \sin(\theta), e^{2(t-t_\alpha^{i,l})} - 1)$$

and we consider the harmonic map $\check{\Phi}_\alpha^{i,l} = \Phi_\alpha \circ (\Psi_\alpha^{i,l})^{-1}$ on S^2 . Let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $0 \leq l \leq t_i$ and $1 \leq i \leq s$. Then,

$$S_\alpha^{i,l} = \Psi_\alpha^{i,l}(P_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, b_\alpha^{i,l} + \theta_\alpha))$$

exhausts S^2 and

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}(\Psi_\alpha^{i,l})_{(\tilde{g}_\alpha)}(S_\alpha^{i,l}) = m_{i,l}.$$

Therefore, we can apply Proposition 5 on the surface (S^2, h) for the sequence $\check{\Phi}_\alpha^{i,l} : S_\alpha^{i,l}, \check{g}_\alpha \rightarrow S^n$. In order to obtain test functions which naturally extend to the manifold, we have to prove that $\mathbf{1}_{S_\alpha^{i,l}} |\nabla \check{\Phi}_\alpha^{i,l}|_h^2 dv_h$ does not concentrate in the poles $(0, 0, 1)$ and $(0, 0, -1)$. By contradiction, if we have

$$\mathbf{1}_{S_\alpha^{i,l}} |\nabla \check{\Phi}_\alpha^{i,l}|_h^2 dv_h \rightharpoonup_* m\delta_{(0,0,1)} + \nu$$

with $m > 0$, $\nu(\{(0, 0, 1)\}) = 0$, then $\int_{S^2} d\nu > 0$ by the hypothesis on $t_\alpha^{i,l}$ we did and up to the extraction of a subsequence, we can build $q_\alpha^{i,l} \ll t_\alpha^{i,l}$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha(a_\alpha^{i,l} - \theta_\alpha, q_\alpha^{i,l})) = m.$$

Setting $\bar{b}_\alpha = q_\alpha^{i,l} + \tau_\alpha$ and $\bar{a}_\alpha = t_\alpha^{i,l} - \tau_\alpha$, with $\tau_\alpha = \sqrt{t_\alpha^{i,l} - r_\alpha^{i,l}}$, we have

$$m_{i,l}^1 = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, \bar{b}_\alpha)) > 0$$

$$m_{i,l}^2 = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(\bar{a}_\alpha, b_\alpha^{i,l} + \theta_\alpha)) > 0$$

with $m_{i,l}^1 + m_{i,l}^2 = m_{i,l}$ and this contradicts the maximality of $\sum_{i=1}^s t_i$.

For fixed $j \in J$, we now focus on the asymptotic behaviour of Φ_α on $M_\alpha^j(c_\alpha)$. We denote by \widetilde{M}_α^j the connected component of $M \setminus (\gamma_\alpha^1, \dots, \gamma_\alpha^s)$ which contains M_α^j . There exists a diffeomorphism $\tau_\alpha : \Sigma_j \rightarrow \widetilde{M}_\alpha^j$ such that (Σ_j, h_α) is a non compact hyperbolic surface with $h_\alpha = \tau_\alpha^* g_\alpha$. On Σ_j , we have

$$h_\alpha \rightarrow h \text{ in } \mathcal{C}_{loc}^\infty(\Sigma_j) \text{ as } \alpha \rightarrow +\infty$$

for a hyperbolic metric h . We let $c = [h]$ and $(\hat{\Sigma}_j, \hat{c})$ the compactification of the cusps of (Σ_j, h) so that $(\hat{\Sigma}_j \setminus \{p_1, \dots, p_t\}, \hat{c})$ is conformal to (Σ_j, c) for some punctures p_1, \dots, p_t as described in [55]. The sequence of sets $\Sigma_\alpha = \tau_\alpha^{-1}(M_\alpha^j(c_\alpha))$ exhausts $\hat{\Sigma}_j$ so that we can apply Proposition 5 on $(\hat{\Sigma}_j, \hat{c})$ to the sequence of harmonic maps $\hat{\Phi}_\alpha = \Phi_\alpha \circ \tau_\alpha : (\Sigma_\alpha, h_\alpha) \rightarrow S^n$. In order to extend on the whole manifold the suitable test functions we define on Σ_j , we will prove that $\mathbf{1}_{\Sigma_\alpha} |\nabla \hat{\Phi}_\alpha|_{h_\alpha}^2 dv_{h_\alpha}$ does not concentrate at the punctures. By contradiction, we assume that

$$\mathbf{1}_{\Sigma_\alpha} |\nabla \hat{\Phi}_\alpha|_{h_\alpha}^2 dv_{h_\alpha} \rightharpoonup_* m\delta_{p_l} + \nu \text{ on } \hat{\Sigma}_j$$

for some puncture $p_l \in \{p_1, \dots, p_t\}$ of $\hat{\Sigma}_j$, with $m > 0$, $\nu(\{p_l\}) = 0$. Then, up to the extraction of a subsequence, we can build $q_\alpha \rightarrow +\infty$ such that

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(-\mu_\alpha^i + q_\alpha, -\mu_\alpha^i + c_\alpha^{i,-})) = m.$$

for $1 \leq i \leq s$ such that $\tau_\alpha^{-1}(\{-\mu_\alpha^i < t < 0\})$ is a neighbourhood of the puncture p_l of $\hat{\Sigma}_j$. We proceed the same way for the symmetric case $\{0 < t < \mu_\alpha^i\}$. Setting $d_\alpha = \sqrt{q_\alpha}$, $\bar{a}_\alpha = -\mu_\alpha^i + q_\alpha - \sqrt{q_\alpha}$ and $\bar{b}_\alpha = -\mu_\alpha^i + c_\alpha^{i,-}$, we have

$$m = \lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha}(P_\alpha^i(\bar{a}_\alpha, \bar{b}_\alpha)) > 0 \text{ and}$$

$$\lim_{\alpha \rightarrow +\infty} \text{Vol}_{\tilde{g}_\alpha} M_\alpha^j(\overline{c}_\alpha) = m_j - m$$

where \overline{c}_α comes from c_α , taking d_α instead of $c_\alpha^{j,-}$. Adding the sequences $\overline{a}_\alpha \ll \overline{b}_\alpha$ contradicts the maximality of $\sum_{i=1}^s t_i$.

As described in Proposition 5 and the computations of section 4.7.2, we can build suitable test functions thanks to the limit functions and associated scales of the sequences $\check{\Phi}_\alpha^{i,l} : S_\alpha^j \subset \mathbb{S}^2 \rightarrow \mathbb{S}^n$ and $\hat{\Phi}_\alpha^j : \Sigma_\alpha \subset \hat{\Sigma}_j \rightarrow \mathbb{S}^n$. They give at least $k+1$ well defined test functions for the variational characterization (4.2.1) of λ_α by the gap assumption of the theorem. Indeed, denoting γ_j the genus of $\hat{\Sigma}_j$, we notice that $\sum_{j \in J} \gamma_j \leq \gamma$ and if $|J| = 1$, $\gamma_1 < \gamma$. These at least $k+1$ test functions give a contradiction. This ends the proof of Theorem 8.

Chapitre 5

Régularité et quantification des applications harmoniques à bord libre

Dans ce travail, en collaboration avec Paul Laurain, nous prouvons un résultat de quantification pour les applications harmoniques à bord libre définies sur une surface Riemannienne arbitraire à valeurs dans la boule unité de \mathbb{R}^{n+1} , à énergie bornée. Nous généralisons des résultats obtenus par Da Lio [24] sur le disque. Nous donnons également une autre démonstration du résultat de régularité de telles applications démontré par Scheven [102].

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5.1 Introduction

Let (M, g) be a smooth Riemannian surface with a smooth nonempty boundary with s connected components. We fix $n \geq 2$ and let \mathbb{B}^{n+1} be the unit ball of \mathbb{R}^{n+1} . A map $u : (M, g) \rightarrow \mathbb{B}^{n+1}$ is a smooth harmonic map with free boundary if it is harmonic and smooth up to the boundary, $u(\partial M) \subset \mathbb{S}^n$ and $\partial_\nu u$ is parallel to u , (or $\partial_\nu u \in (T_u \mathbb{S}^n)^\perp$). The energy of such a map is defined as

$$E(u) = \int_M |\nabla u|_g^2 dv_g = \int_{\partial M} u \cdot \partial_\nu u d\sigma_g.$$

Harmonic maps with free boundary are a natural counterpart of \mathbb{S}^n -valued harmonic maps which are critical points of the energy E with the constraint that $|u|^2 = 1$ on the surface. The difference is that one requires $|u|^2 = 1$ only on the boundary.

Harmonic maps with free boundary naturally appear for instance as critical points of Steklov eigenvalues when the metrics stay in a fixed conformal class, as noticed by Fraser and Schoen [39]. This is the counterpart of \mathbb{S}^n -valued harmonic maps which are critical points for the Laplace eigenvalue when the variation of metrics lie in a fixed conformal class (see for instance Petrides [91]). There are also of course strong links between critical points of Steklov eigenvalues, harmonic maps with free boundary and minimal surfaces in a $(n+1)$ -ball with free boundary conditions. We refer once again to Fraser-Schoen [39].

We prove the following quantification result for harmonic maps with free boundary :

Theorem 12. *Let $u_m : (M, g) \rightarrow \mathbb{B}^{n+1}$ a sequence of harmonic maps with free boundary, i.e $u_m(\partial M) \subset \mathbb{S}^n$ and u_m is parallel to $\partial_\nu u_m$ such that*

$$\limsup_{m \rightarrow +\infty} \int_M |\nabla u_m|_g^2 dv_g < +\infty.$$

Then, there is a harmonic map with free boundary $u_\infty : M \rightarrow \mathbb{B}^{n+1}$ and

- $\omega^1, \dots, \omega^l$ a family of 1/2-harmonic maps $\mathbb{R} \rightarrow \mathbb{S}^n$,
- a_m^1, \dots, a_m^l a family of converging sequences of points on ∂M ,
- $\lambda_m^1, \dots, \lambda_m^l$ a family of sequences of positive numbers all converging to 0,

such that up to the extraction of a subsequence,

$$u_m \rightarrow u_\infty \text{ in } C_{loc}^\infty(M \setminus \{a_\infty^1, \dots, a_\infty^l\}),$$

and

$$\int_{\partial M} R_m \cdot \partial_\nu R_m \rightarrow 0,$$

where we identify ∂M to s copies of $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$:

$$\partial M = \bigcup_{j=1}^s C_j,$$

with $a_\infty^i \in C_j \setminus \{\infty\}$ for some j ,

$$R_m = u_m - u_\infty - \sum_{i=1}^l \omega^i \left(\frac{\cdot - a_m^i}{\lambda_m^i} \right) .$$

In particular, in the space of measures on the boundary, we have the convergence

$$u_m \cdot \partial_\nu u_m d\sigma_g \rightharpoonup_\star u_\infty \cdot \partial_\nu u_\infty d\sigma_g + \sum_{i=1}^l e_i \delta_{a_\infty^i} ,$$

where e_i denotes the energy of the harmonic extension of ω^i on \mathbb{R}_+^2 , noticing that a map ω on the real line $\mathbb{R} \times \{0\}$ is $\frac{1}{2}$ -harmonic if and only if his harmonic extension $\omega : \mathbb{R}_+^2 \rightarrow \mathbb{B}^{n+1}$ is harmonic with free boundary. Notice also that a harmonic map with free boundary and finite energy $\omega : \mathbb{R}_+^2 \rightarrow \mathbb{B}^{n+1}$ corresponds to a harmonic map with free boundary on the disc by Claim 38 below and the remark which follows. Therefore, ω_i is automatically conformal and by Fraser and Schoen [40], we get that $\omega_i(\mathbb{D})$ is an equatorial plane disc and that the energy of such a map satisfies

$$e_i = E(\omega_i) = \int_{\mathbb{R} \times \{0\}} \omega_i \cdot (-\partial_t \omega_i) ds \in 2\pi\mathbb{N} .$$

Thus our result is indeed a quantification result, analogous to those of Sacks-Uhlenbeck [100], Parker [88] for S^n -valued harmonic maps, or of Laurain-Rivière [71] for similar equations.

In the case of the disc, $(M, g) = (\mathbb{D}, \zeta)$, this theorem has already been proved by Da Lio [24]. In this case, the correspondance between harmonic maps with free boundary on the disc with harmonic extensions of $\frac{1}{2}$ -harmonic map on the real line is used. Denoting her energy as

$$E(u) = \int_{\mathbb{R}} \left| \Delta^{\frac{1}{4}} u \right|^2 dx ,$$

she made use of another variational problem on the real line. This correspondance is no more true for general surfaces.

The proof of our theorem relies classically on a ϵ -regularity property. Proving it permits also to prove a regularity result on weakly harmonic maps with free boundary. A map $u : (M, g) \rightarrow \mathbb{B}^{n+1}$ is called weakly harmonic with free boundary if $u(x) \in S^n$ for a.e $x \in \partial M$ and if for any $v \in L^\infty \cap H^1(M, \mathbb{R}^{n+1})$ with $v(x) \in T_{u(x)} S^n$ for a.e $x \in \partial M$,

$$\int_M \langle \nabla u, \nabla v \rangle dv_g = 0 .$$

Such a map is a critical point for the above energy E with respect to the variations $u_t = \pi(u + tv)$ for any $v \in L^\infty \cap H^1(M, \mathbb{R}^{n+1})$ with $v(x) \in T_{u(x)} S^n$ for a.e $x \in \partial M$, where for $z \in \mathbb{R}^{n+1}$, $\pi(z)$ denotes the the nearest point retraction to z in \mathbb{B}^{n+1} . Then we have the following :

Theorem 13 (Scheven [102]). *A weakly harmonic map with a free boundary $u : (M, g) \rightarrow \mathbb{B}^{n+1}$ is always smooth until the boundary and thus is a classical harmonic map with free boundary.*

This regularity theorem was originally proved by Scheven [102], even in a more general context. See also the initial contribution of Duzaar and Steffen [31]. This theorem is the analog of that of Hélein [51] for S^n -valued harmonic maps.

Another proof, in the case of the disc, was also given by Da Lio-Rivière [25], using the correspondence already explained above. Our proof of this result is an adaptation of the proof of Scheven [102]. However, we are more careful and precise in the regularity estimate, passing from a $C^{0,\alpha}$ ϵ -regularity result to a C^1 ϵ -regularity result. This is crucial to prove Theorem 12.

As already said, harmonic maps with free boundary in the unit ball naturally appear as critical points of Steklov eigenvalues. Both our theorems are crucial in order to prove existence of regular metrics which maximize k -th Steklov eigenvalue on a surface, either with a conformal class constraint or not, as was stressed by Fraser and Schoen [41]. This is achieved in Petrides [93].

Our paper is organized as follows : section 5.2 is devoted to the proof of Theorem 13, thanks to an ϵ -regularity result, see Claim 3. We also obtain a crucial result of singularity removability in Claim 4. Our proof is based on the rewriting of the equation with a suitable structure, see Claim 2, permitting to use Wentz's inequality, as studied carefully by Rivière [97]. Section 5.3 is then devoted to the proof of Theorem 12. Capitalizing on the ϵ -regularity result, we are able to prove a no-neck energy result after having described the concentration phenomenon which may appear.

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5.2 Regularity results

We denote by $\mathbb{D}_r(x)$ the Euclidean disc centered at $x \in \mathbb{R}^2$ of radius r and we let $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$. On the upper half space, we let $\mathbb{D}_r^+ = \mathbb{D}_r \cap (\mathbb{R} \times \mathbb{R}_+)$ and $\mathbb{D}^+ = \mathbb{D}_1^+$.

We first recall a lemma proved by Scheven [102], lemma 3.1, which states that a weakly harmonic map with free boundary with small energy cannot vanish close to the boundary ∂M .

Claim 35 ([102]). *There exists ϵ_0 and $C > 0$ such that for any $0 < \epsilon < \epsilon_0$, and any weakly harmonic map $u \in H^1(\mathbb{D}^+, \mathbb{B}^{n+1})$ such that $u(\mathbb{R} \times \{0\}) \subset \mathbb{S}^n$, if*

$$\int_{\mathbb{D}^+} |\nabla u|^2 \leq \epsilon ,$$

then for any $x \in \mathbb{D}_{\frac{1}{2}}^+$,

$$d(u(x), \mathbb{S}^n) \leq C\epsilon^{\frac{1}{2}} .$$

To prove the regularity result, we will extend the weakly harmonic map with free boundary $u : (M, g) \rightarrow \mathbb{B}^{n+1}$ by a symmetrization with respect to the boundary.

Since u is harmonic in the interior of M , it is smooth. It remains to prove that u is smooth at the neighbourhood of each point $a \in \partial M$. We take a conformal chart $\phi : U \rightarrow \mathbb{R}_+^2$ centered at $a \in U$ such that

- $\phi(a) = 0$,
- $\phi^*(e^{2w}\xi) = g$, where w is some smooth function and ξ is the Euclidean metric,
- $\phi(\partial M \cap U) = I = (-1, 1) \times \{0\}$.

We let $\bar{u} = u \circ \phi^{-1}$. Then, by conformal invariance, we have that for any $v \in \mathcal{C}_c^\infty(\mathbb{D}^+, \mathbb{R}^{n+1})$, $v(x) \in T_{\bar{u}(x)}\mathbb{S}^n$ for a.e $x \in (-1, 1) \times \{0\}$,

$$\int_{\mathbb{D}^+} \langle \nabla v, \nabla \bar{u} \rangle = 0.$$

Applying Claim 35, we can assume up to a dilation that

$$\forall x \in \mathbb{D}^+, |\bar{u}(x)| \geq \frac{1}{2}. \quad (5.1)$$

In particular, \bar{u} does not vanish and we can define its extension in \mathbb{D} as follows :

$$\tilde{u} = \begin{cases} \bar{u} & \text{on } \mathbb{D}^+ \\ \sigma \circ \bar{u} \circ r & \text{on } \mathbb{D}^- \end{cases} \quad (5.2)$$

where for $z \in \mathbb{R}^{n+1}$, $\sigma(z) = \frac{z}{|z|^2}$ is the inversion w.r.t. the unit sphere \mathbb{S}^n , and for $x = (s, t) \in \mathbb{R}^2$, $r(x) = (s, -t)$ and $\mathbb{D}^- = r(\mathbb{D}^+)$.

Claim 36. *We have that $\tilde{u} \in H^1(\mathbb{D})$ and satisfies in a weak sense : for $1 \leq j \leq n+1$,*

$$-div(A \nabla \tilde{u}_j) = \sum_{i=1}^{n+1} \langle X_{i,j}, \nabla \tilde{u}_i \rangle \text{ in } \mathbb{D} \quad (5.3)$$

where $A \in H^1(\mathbb{D})$ is defined by

$$A = \begin{cases} 1 & \mathbb{D}^+ \\ \frac{1}{|\bar{u}|^2} & \mathbb{D}^- \end{cases},$$

and for $1 \leq i, j \leq n+1$, $X_{i,j} \in L^2(\mathbb{D})$ is defined by

$$X_{i,j} = \begin{cases} 0 & \mathbb{D}^+ \\ 2 \frac{\tilde{u}_j \nabla \tilde{u}_i - \tilde{u}_i \nabla \tilde{u}_j}{|\tilde{u}|^4} & \mathbb{D}^- \end{cases},$$

and satisfies in a weak sense

$$div(X_{i,j}) = 0 \text{ in } \mathbb{D}. \quad (5.4)$$

Proof. We immediately have that \tilde{u} and $X_{i,j}$ satisfy

$$\Delta \tilde{u} = 0 \text{ and } div(X_{i,j}) = 0 \text{ in } \mathbb{D}^+ \setminus I$$

in a classical sense. Direct computations give that

$$-div\left(\frac{\nabla \tilde{u}}{|\tilde{u}|^4}\right) = 2 \frac{|\nabla \tilde{u}|^2}{|\tilde{u}|^6} \tilde{u} \text{ on } \mathbb{D}^- \setminus I$$

so that

$$div(X_{i,j}) = 0 \text{ in } \mathbb{D}^- \setminus I$$

for $1 \leq i, j \leq n+1$ and

$$-div \left(\frac{\nabla \tilde{u}_j}{|\tilde{u}|^2} \right) = \sum_i \langle X_{i,j}, \nabla \tilde{u}_i \rangle \text{ in } \mathbb{D}^- \setminus I$$

for $1 \leq j \leq n+1$, both in a classical sense. It remains to prove that these equations are still true on \mathbb{D} in a weak sense. Thus, for equation (5.3), we need to prove that

$$\forall v \in \mathcal{C}_c^\infty(\mathbb{D}, \mathbb{R}^{n+1}), \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v \rangle + \int_{\mathbb{D}^-} \left(\frac{\langle \nabla \tilde{u}, \nabla v \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle v_j \right) = 0. \quad (5.5)$$

For that purpose, we first remark that, for any $w \in L^\infty \cap H^1(\mathbb{D}, \mathbb{R}^{n+1})$, we have by direct computations, using the change of variable by the reflection r , that

$$\int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla w \rangle = \int_{\mathbb{D}^-} \langle \nabla (\sigma \circ \tilde{u}), \nabla (w \circ r) \rangle \quad (5.6)$$

since $\sigma \circ \tilde{u} = \tilde{u} \circ r$. More lengthy but straightforward computations also lead to

$$\begin{aligned} & \int_{\mathbb{D}^-} \left(\frac{\langle \nabla \tilde{u}, \nabla w \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle w_j \right) \\ &= \int_{\mathbb{D}^-} \left\langle \nabla (\sigma \circ \tilde{u}), \nabla \left(w - 2 \sum_j \frac{\tilde{u}_j w_j}{|\tilde{u}|^2} \tilde{u} \right) \right\rangle. \end{aligned} \quad (5.7)$$

Indeed,

$$\begin{aligned} & \frac{\langle \nabla \tilde{u}, \nabla w \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle w_j \\ &= \frac{\langle \nabla \tilde{u}, \nabla w \rangle}{|\tilde{u}|^2} - 2 \sum_j \frac{\tilde{u}_j w_j}{|\tilde{u}|^4} |\nabla \tilde{u}|^2 + \sum_j \frac{\langle \nabla \tilde{u}_j, \nabla |\tilde{u}|^2 \rangle w_j}{|\tilde{u}|^4}, \end{aligned} \quad (5.8)$$

$$\nabla (\sigma \circ \tilde{u}) = \nabla \left(\frac{\tilde{u}}{|\tilde{u}|^2} \right) = \frac{\nabla \tilde{u}}{|\tilde{u}|^2} - \frac{\nabla (|\tilde{u}|^2) \tilde{u}}{|\tilde{u}|^4} \quad (5.9)$$

and

$$\begin{aligned} \nabla \left(w - 2 \sum_j \frac{\tilde{u}_j w_j}{|\tilde{u}|^2} \tilde{u} \right) &= \nabla w - 2 \sum_j \frac{\tilde{u}_j w_j}{|\tilde{u}|^2} \nabla \tilde{u} \\ &\quad - 2 \sum_j \frac{\nabla \tilde{u}_j w_j}{|\tilde{u}|^2} \tilde{u} - 2 \sum_j \frac{\nabla w_j \tilde{u}_j}{|\tilde{u}|^2} \tilde{u} + 2 \sum_j \tilde{u}_j w_j \tilde{u} \frac{\nabla |\tilde{u}|^2}{|\tilde{u}|^4}. \end{aligned} \quad (5.10)$$

Let now $v \in \mathcal{C}_c^\infty(\mathbb{D}, \mathbb{R}^{n+1})$ and let us set

$$v_e \circ r = \frac{1}{2} \left(v \circ r + v - 2 (\tilde{u} \cdot v) \frac{\tilde{u}}{|\tilde{u}|^2} \right)$$

and

$$v_a \circ r = \frac{1}{2} \left(v \circ r - v + 2 (\tilde{u} \cdot v) \frac{\tilde{u}}{|\tilde{u}|^2} \right)$$

so that $v_a + v_e = v$. Note that v_a and v_e are in $L^\infty \cap H^1(\mathbb{D}, \mathbb{R}^{n+1})$. Note also that we have

$$v_e \circ r = v_e - 2 \frac{v_e \cdot \tilde{u}}{|\tilde{u}|^2} \tilde{u}$$

and

$$v_a \circ r = - \left(v_a - 2 \frac{v_a \cdot \tilde{u}}{|\tilde{u}|^2} \tilde{u} \right).$$

Then we can write, applying (5.6) and (5.7) with $w = v_a$ and $w = v_e$ and using these last equalities, that

$$\begin{aligned} & \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v \rangle + \int_{\mathbb{D}^-} \left(\frac{\langle \nabla \tilde{u}, \nabla v \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle v_j \right) \\ &= \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_a \rangle + \int_{\mathbb{D}^-} \left(\frac{\langle \nabla \tilde{u}, \nabla v_a \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle (v_a)_j \right) \\ & \quad + \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_e \rangle + \int_{\mathbb{D}^-} \left(\frac{\langle \nabla \tilde{u}, \nabla v_e \rangle}{|\tilde{u}|^2} - \sum_{i,j} \langle X_{i,j}, \nabla \tilde{u}_i \rangle (v_e)_j \right) \\ &= \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_a \rangle - \int_{\mathbb{D}^-} \langle \nabla (\sigma \circ \tilde{u}), \nabla (v_a \circ r) \rangle \\ & \quad + \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_e \rangle + \int_{\mathbb{D}^-} \langle \nabla (\sigma \circ \tilde{u}), \nabla (v_e \circ r) \rangle \\ &= 2 \int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_e \rangle. \end{aligned}$$

Noticing now that if $x \in \mathbb{R} \times \{0\}$,

$$v_e(x) = v(x) - (\tilde{u}(x) \cdot v(x)) \tilde{u}(x) \in T_{\tilde{u}(x)} \mathbb{S}^n,$$

and recalling that u is weakly harmonic with free boundary, we know that

$$\int_{\mathbb{D}^+} \langle \nabla \tilde{u}, \nabla v_e \rangle = 0$$

which clearly ends the proof of (5.5).

It remains to prove (5.4), that is

$$\forall f \in \mathcal{C}_c^\infty(\mathbb{D}, \mathbb{R}^{n+1}), \int_{\mathbb{D}} \langle \nabla f, X_{i,j} \rangle = 0.$$

Let $f \in \mathcal{C}_c^\infty(\mathbb{D}, \mathbb{R})$ and $1 \leq i, j \leq n+1$ be such that $i \neq j$. Then

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{D}} \langle \nabla f, X_{i,j} \rangle &= \int_{\mathbb{D}^-} \left\langle \frac{\tilde{u}_j \nabla \tilde{u}_i - \tilde{u}_i \nabla \tilde{u}_j}{|\tilde{u}|^4}, \nabla f \right\rangle \\ &= \int_{\mathbb{D}^+} \left\langle \frac{(\sigma \circ \tilde{u})_j \nabla (\sigma \circ \tilde{u})_i - (\sigma \circ \tilde{u})_i \nabla (\sigma \circ \tilde{u})_j}{|\tilde{u} \circ \sigma|^4}, \nabla (f \circ r) \right\rangle \\ &= \int_{\mathbb{D}^+} \langle \tilde{u}_j \nabla \tilde{u}_i - \tilde{u}_i \nabla \tilde{u}_j, \nabla (f \circ r) \rangle \\ &= \int_{\mathbb{D}^+} \langle \nabla v_{i,j}, \nabla \tilde{u} \rangle \\ &= 0, \end{aligned}$$

where $v_{i,j} \in C_c^\infty(\mathbb{D}, \mathbb{R}^{n+1})$ is defined by

$$(v_{i,j})_k = f \circ r \times \begin{cases} \tilde{u}_j & \text{if } k = i \\ -\tilde{u}_i & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

and $\langle v_{i,j}(x), \tilde{u}(x) \rangle = 0$ for $x \in \mathbb{D}$. This ends the proof of the claim. \diamond

Notice that this construction is similar to Scheven's one [102]. However, in Claim 36, we give a suitable form to the equation that satisfies the symmetrized map \tilde{u} : we use its structure to prove an ϵ -regularity result that will be useful for the second part of the paper. This type of equations was intensively studied by Rivière (see [97]).

Claim 37. *There is $\epsilon_1 > 0$ and a constant C_k such that if a weakly harmonic map with free boundary u satisfies at the neighbourhood of $a \in \partial M$*

$$\int_{\mathbb{D}^+} |\nabla \tilde{u}|^2 \leq \epsilon_1 ,$$

then $\tilde{u} \in C^\infty(\mathbb{D}_{\frac{1}{2}}^+)$ and for any $k \geq 0$,

$$\|\nabla \tilde{u}\|_{C^k(\mathbb{D}_{\frac{1}{2}}^+)} \leq C_k \|\nabla \tilde{u}\|_{L^2(\mathbb{D}^+)} .$$

Proof. We fix $0 < \epsilon_2 < \epsilon_0$ that we shall choose later, where ϵ_0 is given by Claim 35 and we assume that

$$\int_{\mathbb{D}} |\nabla \tilde{u}|^2 \leq \epsilon_2 .$$

Since $X_{i,j} \in L^2(\mathbb{D})$ satisfies (5.4), that is $\text{div}(X_{i,j}) = 0$ in a weak sense in \mathbb{D} , there is a function $B_{i,j} \in H^1(\mathbb{D})$ such that $X_{i,j} = \nabla^\perp B_{i,j}$. Then, $\tilde{u} \in H^1(\mathbb{D})$ satisfies the equations

$$\begin{cases} -\text{div}(A \nabla \tilde{u}_j) = \sum_i \langle \nabla^\perp B_{i,j}, \nabla \tilde{u}_i \rangle \\ \text{rot}(A \nabla \tilde{u}_i) = \langle \nabla^\perp A, \nabla \tilde{u}_i \rangle \end{cases}$$

Let $p \in \mathbb{D}_{\frac{1}{2}}$ and $0 < r < \frac{1}{2}$. We let $C \in H^1(\mathbb{D})$ be such that

$$\begin{cases} \Delta C_j = \sum_i \langle \nabla^\perp B_{i,j}, \nabla \tilde{u}_i \rangle & \text{in } \mathbb{D}_r(p) \\ C = 0 & \text{on } \partial \mathbb{D}_r(p) \end{cases} \quad (5.11)$$

Since $\text{div}(A \nabla \tilde{u} - \nabla C) = 0$, there is $D \in H^1(\mathbb{D})$ such that $\nabla^\perp D = A \nabla \tilde{u} - \nabla C$. We set $D = \phi + v$ where v is harmonic and ϕ satisfies

$$\begin{cases} \Delta \phi_i = - \langle \nabla^\perp A, \nabla \tilde{u}_i \rangle & \text{in } \mathbb{D}_r(p) \\ \phi = 0 & \text{on } \partial \mathbb{D}_r(p) \end{cases} \quad (5.12)$$

Wente's theorem applied to (5.11) and (5.12) gives the estimates

$$\|\nabla C\|_{L^2(\mathbb{D}_r(p))} \leq K_0 \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^2 \quad \text{and} \quad (5.13)$$

$$\|\nabla \phi\|_{L^2(\mathbb{D}_r(p))} \leq K_0 \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^2 , \quad (5.14)$$

with K_0 a universal constant. Here we used (5.1). Moreover, by Rivière [97], lemma VII.1,

$$\|\nabla v\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))} \leq \frac{1}{16} \|\nabla v\|_{L^2(\mathbb{D}_r(p))} . \quad (5.15)$$

Then we have by Young's inequalities and (5.13), (5.14) that

$$\begin{aligned} \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 &= \|A^{-1}A\nabla \tilde{u}\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 \\ &\leq 2\|A^{-1}\|_\infty^2 \left(\|\nabla C\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 + \|\nabla D\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 \right) \\ &\leq 2\|A^{-1}\|_\infty^2 \left(3K_0^2 \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^4 + 2\|\nabla v\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 \right) . \end{aligned}$$

And an integration by parts on $\mathbb{D}_r(p)$ gives that

$$\|A\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^2 = \|\nabla C\|_{L^2(\mathbb{D}_r(p))}^2 + \|\nabla D\|_{L^2(\mathbb{D}_r(p))}^2$$

since $C = 0$ on $\partial\mathbb{D}_r(p)$. Now we also have that

$$\|\nabla v\|_{L^2(\mathbb{D}_r(p))}^2 = \|\nabla D\|_{L^2(\mathbb{D}_r(p))}^2 - \|\nabla \phi\|_{L^2(\mathbb{D}_r(p))}^2 \leq \|\nabla D\|_{L^2(\mathbb{D}_r(p))}^2$$

so that, since $A(x) \leq 1$ for a.e $x \in \mathbb{D}$, we can write that

$$\|\nabla v\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))} \leq \frac{1}{16} \|\nabla v\|_{L^2(\mathbb{D}_r(p))} \leq \frac{1}{16} \|\nabla D\|_{L^2(\mathbb{D}_r(p))} \leq \frac{1}{16} \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}$$

and we finally get that

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 \leq 6K_0^2 \|A^{-1}\|_\infty^2 \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^4 + \frac{1}{64} \|A^{-1}\|_\infty^2 \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^2 .$$

Since $\|A^{-1}\|_\infty^2 \leq 16$ thanks to (5.1), up to choose $\epsilon_2 = \frac{1}{4 \times 96 K_0^2}$, we get that

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{D}_{\frac{r}{16}}(p))}^2 \leq \frac{1}{2} \|\nabla \tilde{u}\|_{L^2(\mathbb{D}_r(p))}^2$$

for any $p \in \mathbb{D}_{\frac{1}{2}}$ and $0 < r < \frac{1}{2}$. Thanks to Morrey estimates, see page 50 of [97], and the elliptic regularity on the equation, knowing that $|\nabla^\perp B|^2 \leq K|\nabla \tilde{u}|^2$ almost everywhere for some constant K , we get a constant C independent of \tilde{u} such that

$$\|\nabla \tilde{u}\|_{C^{1,\gamma}(\mathbb{D}_{\frac{1}{2}})} \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{D})} . \quad (5.16)$$

Since by (5.2),

$$\int_{\mathbb{D}^-} |\nabla \tilde{u}|^2 = \int_{\mathbb{D}^+} \frac{|\nabla \tilde{u}|^2}{|\tilde{u}|^4} ,$$

using (5.1), and setting $\epsilon_1 = \frac{\epsilon_2}{17}$, we get

$$\int_{\mathbb{D}^+} |\nabla \tilde{u}|^2 \leq \epsilon_1 \Rightarrow \int_{\mathbb{D}} |\nabla \tilde{u}|^2 \leq \epsilon_2 .$$

Finally, by elliptic regularity theory, we can bootstrap (5.16) thanks to the equation on the half space,

$$\begin{cases} \Delta \tilde{u} = 0 & \text{on } \mathbb{D}^+ \\ -\partial_t \tilde{u} = (\tilde{u} \cdot (-\partial_t \tilde{u})) \tilde{u} & \text{on } I \end{cases}$$

and get the claim. \diamond

Theorem 13 of course follows from this claim.

Thanks to Claim 37, we also have a result of removability of singularities for harmonic maps with free boundary which will be useful in the next section.

Claim 38. *Let $u : M \setminus \{a\} \rightarrow \mathbb{B}^{n+1}$ with finite energy be such that for any $v \in H^1(M, \mathbb{R}^{n+1}) \cap L^\infty$, $\text{supp}(v) \subset M \setminus \{a\}$ and $v(x) \in T_{u(x)}\mathbb{S}^n$ for a.e $x \in \partial M$,*

$$\int_M \langle \nabla u, \nabla v \rangle_g dv_g = 0.$$

Then u extends to a harmonic map with free boundary $\tilde{u} : M \rightarrow \mathbb{B}^{n+1}$.

Proof. First, using Claim 37, it is clear that u is smooth outside of a . We use in the sequel the same notation as above since the problem is purely local. Thus we can assume that $M = \mathbb{D}^+$, that the metric is Euclidean and that $a = 0$. By a direct scaling argument, using again Claim 37 and standard elliptic regularity theory for harmonic maps in the inside of \mathbb{D}^+ , there exists a constant $C > 0$ such that

$$\sup_{\mathbb{D}_{\frac{|p|}{4}}(p) \cap \mathbb{D}^+} |x|^2 |\nabla u|^2 \leq C \int_{\mathbb{D}_{\frac{|p|}{2}}(p) \cap \mathbb{D}^+} |\nabla u|^2$$

for all $p \in \mathbb{D}^+$ with $|p| \leq \frac{1}{2}$ as soon as

$$\int_{\mathbb{D}_{\frac{|p|}{2}}(p) \cap \mathbb{D}^+} |\nabla u|^2 \leq \epsilon_1.$$

Since $\nabla u \in L^2(\mathbb{D}^+)$ by assumption, we deduce that

$$\sup_{\mathbb{D}_r^+} |x| |\nabla u| \rightarrow 0 \text{ as } r \rightarrow 0. \quad (5.17)$$

Let $v \in C_c^\infty(\mathbb{D}^+)$ be such that for all $x \in \mathbb{R} \times \{0\}$, $v(x) \in T_{u(x)}\mathbb{S}^n$. Then we have, integrating by parts, that

$$\int_{\mathbb{D}^+ \setminus \mathbb{D}_r^+} \langle \nabla u, \nabla v \rangle = \int_{\partial \mathbb{D}_r^+} -\partial_\nu u \cdot v.$$

Using (5.17) and the fact that $\nabla u \in L^2(\mathbb{D}^+)$, we can pass to the limit as $r \rightarrow 0$ to obtain that u is in fact a weak harmonic map with free boundary. It is thus regular thanks to Theorem 13 we just proved. \diamond

Notice that thanks to Claim 38, we have a correspondence between harmonic maps with free boundary $u : \mathbb{D} \rightarrow \mathbb{B}^{n+1}$ and $v : \mathbb{R}_+^2 \rightarrow \mathbb{B}^{n+1}$, thanks to $f : \mathbb{R}_+^2 \rightarrow \mathbb{D} \setminus \{(0,1)\}$, the conformal map defined by $f(z) = \frac{z-i}{z+i}$.

Finally, Claim 37 reveals an energy gap for harmonic maps with free boundary on discs : if a harmonic map with free boundary $\omega : \mathbb{R}_+^2 \rightarrow \mathbb{B}^{n+1}$ satisfies

$$\int_{\mathbb{R}_+^2} |\nabla \omega|^2 \leq \epsilon_1$$

then ω is a constant map. Indeed, by Claim 37 and an obvious scaling argument, we get that

$$\|\nabla \omega\|_{C^0(\mathbb{D}_R^+)} \leq \frac{C_0}{R} \|\nabla \omega\|_{L^2}$$

for all $R > 0$ for some fixed constant C_0 . Letting R go to $+\infty$ gives that ω is constant.

5.3 The quantification phenomenon

We aim at proving Theorem 12.

Step 1 : Points of concentration.

Since the energy of the sequence (u_m) of harmonic maps with free boundary is bounded, we only have a finite number of points, denoted by a^1, \dots, a^q such that

$$\forall r > 0, \limsup_{m \rightarrow +\infty} \int_{B_g(a^i, r)} |\nabla u_m|_g^2 dv_g > \epsilon_1. \quad (5.18)$$

Notice that $a^i \in \partial M$ for any $1 \leq i \leq q$. Indeed, if $a^i \in M \setminus \partial M$, then, since u_m is harmonic, elliptic regularity theory gives a constant C independent of m such that

$$\|\nabla u_m\|_{C^0(B_g(a^i, \frac{\delta}{2}))} \leq C \|u_m\|_{W^{1,2}},$$

where $\delta = d(a^i, \partial M) > 0$, so that (∇u_m) is uniformly bounded on $B_g(a^i, \frac{\delta}{2})$. This contradicts (5.18).

By ϵ -regularity around each point of $\partial M \setminus \{a^1, \dots, a^q\}$, (see Claim 37), we get that

$$u_m \rightarrow u_\infty \text{ in } \mathcal{C}_{loc}^1(M \setminus \{a^1, \dots, a^q\}) \text{ as } m \rightarrow +\infty, \quad (5.19)$$

where u_∞ satisfies the hypothesis of Claim 38 so that u_∞ extends to an harmonic map with free boundary in $\mathcal{C}^\infty(M, \mathbb{R}^{n+1})$.

Step 2 : Blow-up around $a^i \in \partial M$.

We take a conformal chart $\phi_i : U_i \rightarrow \mathbb{R}_+^2$ centered at $a^i \in U_i$ such that

- $\phi_i(a^i) = 0$,
- $\phi_i^*(e^{2w_i}\zeta) = g$, where w_i is some smooth function and ζ is the Euclidean metric,
- $\phi_i(\partial M \cap U_i) \subset \mathbb{R} \times \{0\}$.

We fix $1 \leq i \leq q$ and we let $\bar{u}_m^i = u_m \circ \phi_i^{-1}$ and $\bar{u}_\infty^i = u_\infty \circ \phi_i^{-1}$. We choose $r_i > 0$ small enough such that for any $j \neq i$, $\phi_i(a^j) \notin \mathbb{D}_{r_i}^+$ and at the neighbourhood of $a^i \in \partial M$,

$$\int_{\mathbb{D}_{r_i}^+} |\nabla \bar{u}_\infty^i|^2 < \frac{\epsilon_1}{4}. \quad (5.20)$$

Since a^i satisfies (5.18), we can take λ_m^i such that

$$\int_{\mathbb{D}_{r_i}^+ \setminus \mathbb{D}_{\lambda_m^i}^+} |\nabla \bar{u}_m^i|^2 = \frac{\epsilon_1}{2}. \quad (5.21)$$

Notice that

$$\lambda_m^i \rightarrow 0 \text{ as } m \rightarrow +\infty. \quad (5.22)$$

Indeed, if $\limsup_{m \rightarrow +\infty} \lambda_m^i > 0$, by the definition of r_i , and (5.19), passing to the limit in (5.21) would contradict (5.20).

We set for $x \in \mathbb{R}_+^2$

$$\tilde{u}_m^i(x) = \tilde{u}_m^i(\lambda_m^i x).$$

Since \tilde{u}_m^i is harmonic with finite energy, elliptic estimates prove that, \tilde{u}_m^i does not concentrate on $\mathbb{R} \times (0, +\infty)$. Moreover, by (5.21) and Claim 37, \tilde{u}_m^i does not concentrate on $\mathbb{R}_+^2 \setminus (-1, 1) \times \{0\}$. Therefore, outside some concentration points $a^{i,1}, \dots, a^{i,q_i} \in (-1, 1) \times \{0\}$, we have

$$\tilde{u}_m^i \rightarrow \tilde{u}_\infty^i \text{ in } \mathcal{C}_{loc}^1(\mathbb{R}_+^2 \setminus \{a^{i,1}, \dots, a^{i,q_i}\}) \text{ as } m \rightarrow +\infty. \quad (5.23)$$

Let $f : \mathbb{R}_+^2 \rightarrow \mathbb{D}$ the conformal map defined by $f(z) = \frac{z-i}{z+i}$. Then $\tilde{u}_\infty^i \circ f^{-1}$ satisfies the hypotheses of Claim 38 on $\mathbb{D} \setminus \{1, f(a^{i,1}), \dots, f(a^{i,q_i})\}$ so that $\tilde{u}_\infty^i \circ f^{-1}$ extends to an harmonic map with free boundary in $\mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^{n+1})$. Thanks to (5.21), (5.22) and Claim 39, we have that

$$\lim_{R \rightarrow +\infty} \lim_{m \rightarrow +\infty} \int_{\mathbb{D}_R^+ \setminus \mathbb{D}_{\frac{r_i}{\lambda_m^i}}^+} |\nabla \tilde{u}_m^i|^2 = 0$$

so that by (5.21) and (5.23) for R large enough,

$$\int_{\mathbb{D}_R^+ \setminus \mathbb{D}^+} |\nabla \tilde{u}_\infty^i|^2 > \frac{\epsilon_1}{4}. \quad (5.24)$$

In particular \tilde{u}_∞^i is a non-constant function, which is a $\frac{1}{2}$ -harmonic map on the boundary (one of the ω^j 's given by Theorem 12).

Step 3 : Iteration.

As a classical bubble tree extraction (see [88]), we have two cases : Either there are concentration points and we go back to Step 2 at the neighbourhood of each $a^{i,j}$. Or there is no concentration points for the sequence (\tilde{u}_m^i) (that is $q_i = 0$ in Step 2) and the process stops.

This process has to stop since at every new concentration point, we get a bubble whose energy is at least $\frac{\epsilon_1}{4}$ by (5.24).

Finally, we state a no-neck-energy lemma which concludes the proof of Theorem 12.

Claim 39. *Let (λ_m) be a sequence of positive numbers converging to 0. Let (u_m) be a sequence of harmonic maps on \mathbb{D}^+ with uniformly bounded energy and free boundary on $(-1, 1) \times \{0\}$ such that*

$$\int_{\mathbb{D}^+ \setminus \mathbb{D}_{\lambda_m}^+} |\nabla u_m|^2 \leq \frac{\epsilon_1}{2}, \quad (5.25)$$

Then,

$$\lim_{R \rightarrow +\infty} \lim_{m \rightarrow +\infty} \int_{\mathbb{D}_{\frac{1}{R}}^+ \setminus \mathbb{D}_{\lambda_m R}^+} |\nabla u_m|^2 = 0 \quad (5.26)$$

and

$$\lim_{R \rightarrow +\infty} \lim_{m \rightarrow +\infty} \int_{(-\frac{1}{R}, \frac{1}{R}) \setminus (-\lambda_m R, \lambda_m R)} u_m \cdot \partial_t u_m = 0. \quad (5.27)$$

Proof.

We set $\mathbb{A}_{m,R} = \mathbb{D}_{\frac{1}{R}}^+ \setminus \mathbb{D}_{\lambda_m R}^+$, $I_{m,R} = \mathbb{A}_{m,R} \cap (\mathbb{R} \times \{0\})$ and

$$\delta_{m,R} = \max_{z \in \mathbb{A}_{m,R}} |z| |\nabla u_m| (z) .$$

Step 1 : We have that

$$\lim_{R \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \delta_{m,R} = 0 . \quad (5.28)$$

Proof of Step 1 - Notice that $R \mapsto \delta_{m,R}$ and $R \mapsto \limsup_{m \rightarrow +\infty} \delta_{m,R}$ are nonincreasing. We proceed by contradiction, assuming (5.28) is false. Then there exists a subsequence $(m_\alpha)_{\alpha \geq 1}$ converging to $+\infty$ such that

$$\delta_{m_\alpha, \alpha} \geq \epsilon_0 > 0 \quad (5.29)$$

for some $\epsilon_0 > 0$ fixed. Let $z_\alpha \in \mathbb{A}_{m_\alpha, \alpha}$ be such that $\delta_{m_\alpha, \alpha} = |z_\alpha| |\nabla u_{m_\alpha}| (z_\alpha)$. It is clear that $z_\alpha \rightarrow 0$ and $\frac{|z_\alpha|}{\lambda_{m_\alpha}} \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We let $\bar{u}_\alpha(x) = u_{m_\alpha}(|z_\alpha| x)$ so that, by Claim 37,

$$\bar{u}_\alpha \rightarrow \bar{u}_\infty \text{ in } C_{loc}^1(\mathbb{R}_+^2 \setminus \{0\}) \text{ as } \alpha \rightarrow +\infty$$

where \bar{u}_∞ is harmonic with free boundary. Then, since, after passing to a subsequence,

$$|z_\alpha| |\nabla u_{m_\alpha}| (z_\alpha) = \left| \nabla \bar{u}_\alpha \left(\frac{z_\alpha}{|z_\alpha|} \right) \right| \rightarrow |\nabla \bar{u}_\infty(z)| \text{ as } \alpha \rightarrow +\infty$$

where $z = \lim_{\alpha \rightarrow +\infty} \frac{z_\alpha}{|z_\alpha|}$, we get thanks to (5.29) that $|\nabla \bar{u}_\infty(z)| \geq \epsilon_0$. By assumption (5.25), $\|\nabla \bar{u}_\infty\|_{L^2(\mathbb{R}_+^2)}^2 \leq \frac{\epsilon_1}{2}$ so that by the remark at the end of Section 1, \bar{u}_∞ should be constant. This is a contradiction which ends the proof of this step.

Step 2 :

$$\lim_{R \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \|\nabla u_m\|_{L^{2,\infty}(\mathbb{A}_{m,R})} = 0 \quad (5.30)$$

Proof of Step 2 - We easily check that $\left\| \frac{1}{|x|} \right\|_{L^{2,\infty}(\mathbb{A}_{m,R})} \leq \sqrt{\pi}$ for any m and R , so that

$$\|\nabla u_m\|_{L^{2,\infty}(\mathbb{A}_{m,R})} \leq \sqrt{\pi} \delta_{m,R}$$

and we get (5.30) thanks to Step 1.

Step 3 :

$$\limsup_{R \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \|\nabla_\theta u_m\|_{L^{2,1}(\mathbb{A}_{m,R})} < +\infty \quad (5.31)$$

Proof of Step 3 - Here we use the symmetrization process given by Claim 36. And such estimates on the angular derivative for solutions of this type of equations were obtained in Laurain-Rivi re [71]. First, we have to ensure that $|u_m(x)|$ does not vanish for $x \in \mathbb{A}_{m,R}$. For $x \in \mathbb{A}_{m,R}$, we have that

$$|u_m(x) - u_m(y)| \leq \frac{\pi}{2} |x| \sup_{|z|=|x|} |\nabla u_m|$$

for some $y \in (-1, 1) \times \{0\}$. Since $|u_m(y)| = 1$, we deduce that

$$|u_m(x)| \geq 1 - \frac{\pi}{2} \delta_{m,R}$$

Since $R \mapsto \delta_{m,R}$ are decreasing for every m and thanks to (5.28), we deduce that there exists R_0 and m_0 such that for all $R \geq R_0$ and all $m \geq m_0$,

$$|u_m| \geq \frac{1}{2} \text{ in } \mathbb{A}_{m,R}.$$

Up to a scaling and since we are interested only in large m and large R and in order to simplify the notations, we may assume that $R_0 = 1$ and that $m_0 = 1$. We set now

$$\tilde{u}_m = \begin{cases} u_m & \text{in } \mathbb{D}^+ \setminus \mathbb{D}_{\lambda_m}^+ \\ \sigma \circ u_m \circ r & \text{in } \mathbb{D}^- \setminus \mathbb{D}_{\lambda_m}^- \end{cases} \quad (5.32)$$

Applying the computations of Claim 36, we get that, for $1 \leq j \leq n+1$,

$$-div(A_m \nabla \tilde{u}_m^j) = \sum_{i=1}^{n+1} \langle X_m^{i,j}, \nabla \tilde{u}_m^i \rangle \quad (5.33)$$

in a weak sense where A_m is defined by

$$A_m = \begin{cases} 1 & \mathbb{D}^+ \setminus \mathbb{D}_{\lambda_m}^+ \\ \frac{1}{|\tilde{u}_m|^2} & \mathbb{D}^- \setminus \mathbb{D}_{\lambda_m}^- \end{cases},$$

and for $1 \leq i, j \leq n+1$, $X_m^{i,j}$ is defined by

$$X_m^{i,j} = \begin{cases} 0 & \mathbb{D}^+ \setminus \mathbb{D}_{\lambda_m}^+ \\ 2 \frac{\tilde{u}_m^j \nabla \tilde{u}_m^i - \tilde{u}_m^i \nabla \tilde{u}_m^j}{|\tilde{u}_m|^4} & \mathbb{D}^- \setminus \mathbb{D}_{\lambda_m}^- \end{cases},$$

and satisfies in a weak sense

$$div(X_m^{i,j}) = 0. \quad (5.34)$$

We also have that for $\lambda_m < r < 1$,

$$\begin{aligned} \frac{1}{2} \int_{\partial \mathbb{D}_r} \langle X_m^{i,j}, \nu \rangle &= \int_{\partial \mathbb{D}_r^-} \left\langle \frac{\tilde{u}_m^j \nabla \tilde{u}_m^i - \tilde{u}_m^i \nabla \tilde{u}_m^j}{|\tilde{u}_m|^4}, \nu \right\rangle \\ &= \int_{\partial \mathbb{D}_r^+} \left\langle \frac{(\sigma \circ u_m)^j \nabla (\sigma \circ u_m)^i - (\sigma \circ u_m)^i \nabla (\sigma \circ u_m)^j}{|\sigma \circ u_m|^4}, \nu \right\rangle \\ &= \int_{\partial \mathbb{D}_r^+} u_m^j \partial_\nu u_m^i - u_m^i \partial_\nu u_m^j \\ &= \int_{[-r,r] \times \{0\}} (u_m^j \partial_t u_m^i - u_m^i \partial_t u_m^j) \\ &\quad - \int_{\mathbb{D}_r^+} div(u_m^j \nabla u_m^i - u_m^i \nabla u_m^j) \\ &= 0, \end{aligned}$$

since $\Delta u_m = 0$ and $\partial_t u_m$ is parallel to u_m on $[-r, r] \times \{0\}$. From this and (5.34), we deduce that there exists $B_m^{i,j}$ such that $X_m^{i,j} = \nabla^\perp B_m^{i,j}$. Now, we still denote by \tilde{u}_m and A_m extensions of \tilde{u}_m and A_m on $L^\infty \cap H^1(\mathbb{D})$ such that there is a constant D independent of m with

$$\|\nabla \tilde{u}_m\|_{L^2(\mathbb{D})} \leq D \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})} \quad \text{and} \quad \|\nabla A_m\|_{L^2(\mathbb{D})} \leq D \|\nabla A_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})} . \quad (5.35)$$

For instance, if $f : \mathbb{D} \setminus \mathbb{D}_{\lambda_m} \rightarrow \mathbb{R}$, we take the extension

$$f(z) = f\left(\frac{z\lambda_m^2}{|z|^2}\right) \phi\left(\frac{z\lambda_m^2}{|z|^2}\right)$$

for $z \in \mathbb{D}_{\lambda_m}$, where $\phi \in \mathcal{C}_c^\infty(\mathbb{D})$ is a cut-off function such that $\phi = 1$ on \mathbb{D}_{λ_m} .

We let D_m be such that

$$\begin{cases} \Delta D_m^i = \langle \nabla^\perp A_m, \nabla \tilde{u}_m^i \rangle & \text{in } \mathbb{D} \\ D_m = 0 & \text{on } \partial\mathbb{D} \end{cases} \quad (5.36)$$

Since $\text{rot}(A_m \nabla \tilde{u}_m - \nabla^\perp D_m) = 0$, there is C_m such that $\nabla C_m = A_m \nabla \tilde{u}_m - \nabla^\perp D_m$, and C_m satisfies the equation

$$\Delta C_m^j = \sum_i \langle \nabla^\perp B_m^{i,j}, \nabla \tilde{u}_m^i \rangle$$

on $\mathbb{D} \setminus \mathbb{D}_{\lambda_m}$. We set $C_m = \psi_m + v_m$ where v_m is harmonic and ψ_m satisfies

$$\begin{cases} \Delta \psi_m^j = \sum_i \langle \nabla^\perp B_m^{i,j}, \nabla \tilde{u}_m^i \rangle & \text{in } \mathbb{D} \setminus \mathbb{D}_{\lambda_m} \\ \psi_m = 0 & \text{on } \partial\mathbb{D} \cup \partial\mathbb{D}_{\lambda_m} . \end{cases} \quad (5.37)$$

Wente's estimates involving the $L^{2,1}$ -norm on the disc for (5.36) with the estimates (5.35) on the extensions \tilde{u}_m and A_m and $L^{2,1}$ -Wente's estimates on the annulus (see [71], lemma 2.1) for (5.37) give constants K_0 and K_1 independent of m such that

$$\|\nabla D_m\|_{L^{2,1}(\mathbb{D})} \leq K_0 \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2 \quad (5.38)$$

$$\|\nabla \psi_m\|_{L^{2,1}(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})} \leq K_1 \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2 . \quad (5.39)$$

Since v_m is harmonic, we get a Fourier series

$$v_m = c_m^0 + d_m^0 \ln(r) + \sum_{p \in \mathbb{Z}^*} (c_m^p r^p + d_m^p r^{-p}) e^{ip\theta}$$

and since $\nabla_\theta v_m$ has no logarithm part, we use [71], lemma A.2, to get a constant K_2 independent of m such that

$$\|\nabla_\theta v_m\|_{L^{2,1}(\mathbb{D}_{\frac{1}{2}} \setminus \mathbb{D}_{2\lambda_m})} \leq K_2 \|\nabla v_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})} \leq K_2 \|\nabla C_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}$$

since v_m is the harmonic extension of C_m on $\mathbb{D} \setminus \mathbb{D}_{\lambda_m}$. Then, by Young inequalities and since $A_m \leq 1$,

$$\|\nabla_\theta v_m\|_{L^{2,1}(\mathbb{D}_{\frac{1}{2}} \setminus \mathbb{D}_{2\lambda_m})}^2 \leq 2K_2^2 \left(\|\nabla D_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2 + \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2 \right) \quad (5.40)$$

Now, (5.38), (5.39) and (5.40) give that

$$\begin{aligned} \|\nabla_{\theta} \tilde{u}_m\|_{L^{2,1}(\mathbb{D}_{\frac{1}{2}} \setminus \mathbb{D}_{2\lambda_m})} &\leq 4(K_0 + K_1) \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2 \\ &\quad + 4\sqrt{2}K_2 \sqrt{K_0^2 \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^4 + \|\nabla \tilde{u}_m\|_{L^2(\mathbb{D} \setminus \mathbb{D}_{\lambda_m})}^2} . \end{aligned}$$

Looking at this inequality in $\mathbb{A}_{m,R}$ completes the proof of Step 3.

Gathering Step 2 and Step 3, the duality $L^{2,1} - L^{2,\infty}$ gives that

$$\lim_{R \rightarrow +\infty} \lim_{m \rightarrow +\infty} \|\nabla_{\theta} u_m\|_{L^2(\mathbb{A}_{m,R})} = 0 . \quad (5.41)$$

Since u_m is harmonic with free boundary in \mathbb{D}^+ , we have the following Pohozaev identity

$$\int_{\partial \mathbb{D}_r^+} |\nabla_{\theta} u_m|^2 = \int_{\partial \mathbb{D}_r^+} |\nabla_r u_m|^2 \text{ for all } 0 < r < 1 . \quad (5.42)$$

Indeed, let us write with some integration by parts that

$$\begin{aligned} 0 &= \int_{\mathbb{D}_r^+} \left(x (u_m)_x + y (u_m)_y \right) \cdot \Delta u_m \\ &= -r \int_{\partial \mathbb{D}_r^+} |\nabla_r u_m|^2 + \int_{(-1,1) \times \{0\}} \left(x (u_m)_x + y (u_m)_y \right) \cdot (u_m)_y \\ &\quad + \int_{\mathbb{D}_r^+} |\nabla u_m|^2 + \frac{1}{2} \int_{\mathbb{D}_r^+} \left(x \left(|\nabla u_m|^2 \right)_x + y \left(|\nabla u_m|^2 \right)_y \right) \\ &= -r \int_{\partial \mathbb{D}_r^+} |\nabla_r u_m|^2 + \int_{(-1,1) \times \{0\}} \left(x (u_m)_x + y (u_m)_y \right) \cdot (u_m)_y + \frac{1}{2} r \int_{\partial \mathbb{D}_r^+} |\nabla u_m|^2 . \end{aligned}$$

Now we have that

$$\int_{(-1,1) \times \{0\}} \left(x (u_m)_x + y (u_m)_y \right) \cdot (u_m)_y = 0$$

since on $(-1,1) \times \{0\}$,

$$\left(x (u_m)_x + y (u_m)_y \right) \in T_u \mathbb{S}^n$$

and $(u_m)_y$ is orthogonal to $T_u \mathbb{S}^n$. This proves (5.42) since $|\nabla u_m|^2 = |\nabla_r u_m|^2 + |\nabla_{\theta} u_m|^2$. Integrating (5.42) gives that

$$\int_{\mathbb{A}_{m,R}} |\nabla u_m|^2 = 2 \int_{\mathbb{A}_{m,R}} |\nabla_{\theta} u_m|^2$$

so that (5.26) follows thanks to (5.41).

Finally, we have that

$$\int_{I_{m,R}} u_m \cdot (-\partial_t u_m) = \int_{\mathbb{A}_{m,R}} |\nabla u_m|^2 - \int_{\partial \mathbb{D}_{\frac{1}{R}}^+} u_m \cdot \partial_r u_m + \int_{\partial \mathbb{D}_{R\lambda_m}^+} u_m \cdot \partial_r u_m$$

so that

$$\left| \int_{I_{m,R}} u_m \cdot (-\partial_t u_m) \right| \leq \int_{\mathbb{A}_{m,R}} |\nabla u_m|^2 + \int_{\partial \mathbb{D}_{\frac{1}{R}}^+} |\nabla u_m| + \int_{\partial \mathbb{D}_{R\lambda_m}^+} |\nabla u_m|$$

and (5.27) follows from (5.26) and (5.28).

This ends the proof of Claim 5, and as already said finishes the proof of Theorem 12. \diamond

Chapitre 6

Maximiser les valeurs propres de Steklov sur une surface

Nous étudions les fonctionnelles valeurs propres de Steklov $\sigma_k(\Sigma, g) L_g(\partial\Sigma)$ sur des surfaces à bord non vide (Σ, g) . Nous démontrons que sous certaines hypothèses naturelles, ces fonctionnelles admettent des métriques maximales qui viennent avec une surface minimale à bord libre de Σ dans une boule euclidienne.

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6.1 Introduction

Let Σ be a smooth compact connected surface with a smooth boundary $\partial\Sigma \neq \emptyset$. We denote by γ its genus and by m the number of connected components of its boundary, which, together with orientability, characterize topologically the surface. Given a Riemannian metric g on Σ , the Dirichlet-to-Neumann operator, $L : C^\infty(\partial\Sigma) \mapsto C^\infty(\partial\Sigma)$, is defined as follows : for any $u \in C^\infty(\partial\Sigma)$, consider the harmonic extension \hat{u} of u in Σ , which is unique, then $Lu = \partial_\nu \hat{u}$ where ν is the outward unit conormal along $\partial\Sigma$. This operator is self-adjoint and has a discrete spectrum

$$0 = \sigma_0 < \sigma_1(\Sigma, g) \leq \sigma_2(\Sigma, g) \leq \dots \leq \sigma_k(\Sigma, g) \leq \dots \rightarrow +\infty$$

of so-called Steklov eigenvalues counted with multiplicity. These are the σ 's for which there exists a non-trivial solution $u \in C^\infty(\Sigma)$, smooth up to the boundary, of

$$\begin{cases} \Delta_g u = 0 & \text{in } \Sigma \\ \partial_\nu u = \sigma u & \text{on } \partial\Sigma \end{cases}$$

where $\Delta_g = -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator. These eigenvalues are also characterized by the following variational problem :

$$\sigma_k(\Sigma, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_\Sigma |\nabla \phi|_g^2 dv_g}{\int_{\partial\Sigma} \phi^2 d\sigma_g},$$

where the infimum is taken over the vector spaces of smooth functions E_{k+1} of dimension $k+1$.

These eigenvalues may be seen as functionals depending on the metric g . For obvious scaling reasons, it is more interesting to consider the functionals $\sigma_k(\Sigma, g) L_g(\partial\Sigma)$. There has been a recent interest in studying these Steklov eigenvalue functionals because of the connection between critical metrics for these functionals and minimal immersions of Σ with free boundary into some Euclidean ball. A smooth immersion $\Phi : \Sigma \mapsto \mathbb{B}^{n+1}$ is a minimal surface with free boundary if $\Phi(\Sigma)$ is a minimal surface with $\Phi(\partial\Sigma) \subset \mathbb{S}^n$ which hits the boundary orthogonally (that is, $\partial_\nu \Phi$ is parallel to Φ on $\partial\Sigma$). These free boundary minimal surfaces arise as critical points of the area when the surface is constrained to lie in the ball but is free to vary on the boundary of the ball. This link between this purely geometric problem and the Steklov eigenvalues was first discovered by Fraser-Schoen [38]. In particular, it is proved in Fraser-Schoen [39], proposition 2.4, that a metric g_0 on Σ such that $\sigma_k(\Sigma, g_0) L_{g_0}(\partial\Sigma)$ is maximal among smooth metrics on Σ comes with a conformal minimal immersion with free boundary $\Phi : \Sigma \mapsto \mathbb{B}^{n+1}$ for some n such that Φ is an isometry on $\partial\Sigma$, up to scaling. Note that, conversely, see again Fraser-Schoen [39], the coordinates of any conformal minimal immersion with free boundary are Steklov eigenfunctions corresponding to some σ_k . This link has led Fraser and Schoen to start an intensive study of the first Steklov eigenvalue (see [38], [39], [41]).

Thus it is geometrically interesting to look for maximal metrics for Steklov eigenvalues in order to get conformal minimal immersions with free boundary. That's a good reason to introduce the topological invariant

$$\sigma_k(\gamma, m) = \sup_g \sigma_k(\Sigma, g) L_g(\partial\Sigma)$$

where Σ is an oriented surface of genus γ with m boundary components. Girouard and Polterovich [47] proved that

$$\sigma_k(\gamma, m) \leq 2\pi k(\gamma + m) ,$$

generalizing for $k \geq 2$ an estimate due to Fraser and Schoen [38] in the case $k = 1$. Very few exact values of $\sigma_k(\gamma, m)$ are known. Weinstock [112] proved in 1954 that

$$\sigma_k(0, 1) = 2\pi k \tag{6.1}$$

and that for $k = 1$, the case of equality holds for the Euclidean disc. The exact value of $\sigma_1(0, 2)$ was found by Fraser-Schoen [41] and the maximizing metric was characterized as coming from the critical catenoid. In this same paper, an asymptotic of $\sigma_1(0, m)$ as $m \rightarrow +\infty$ was obtained.

It can also be shown by standard gluing procedures (even if a bit technical, see [30]) that the following inequalities between these topological invariants hold :

$$\sigma_k(\gamma, m) \geq \max_{\substack{i_1 + \dots + i_s = k \\ \forall q, i_q \geq 1 \\ \gamma_1 + \dots + \gamma_s \leq \gamma \\ m_1 + \dots + m_s \leq m \\ \gamma_1 < \gamma \text{ or } m_1 < m \text{ if } s=1}} \sum_{q=1}^s \sigma_{i_q}(\gamma_q, m_q) . \tag{6.2}$$

We prove the following existence result :

Theorem 14. *Let Σ be a compact orientable surface of genus γ , with a smooth boundary with $m \geq 1$ connected components. Let $k \geq 1$. If the inequality (6.2) is strict, then there exists a smooth metric g on Σ such that $\sigma_k(\gamma, m) = \sigma_k(\Sigma, g) L_g(\partial\Sigma)$. Moreover, up to scaling, this maximizing metric is the pull-back of the Euclidean metric by some conformal minimal immersion with free boundary in the unit Euclidean ball \mathbb{B}^{n+1} for some n .*

This theorem was proved for the first eigenvalue $k = 1$, with $\gamma = 0$ and any m in Fraser-Schoen [41]. In this case, the condition that (6.2) is strict reads as $\sigma_1(0, m) > \sigma_1(0, m-1)$. They also proved that this condition holds true for any m so that $\sigma_1(0, m)$ is achieved by a smooth maximal metric for all $m \geq 1$. Their proof easily extends to higher genus, still for $k = 1$, except that we do not know if the gap condition holds for $\gamma \geq 1$.

Note that our theorem gives suitable conditions for the existence of conformal minimal immersion with free boundary of various genus and number of boundary components given by k -th Steklov eigenfunctions for any $k \geq 1$. Note also that the gap assumption, i.e. the fact that (6.2) is strict, is necessary to get an existence result. Indeed, it was proved by Girouard-Polterovich [44] that $\sigma_2(0, 1)$ is not achieved by a maximizing metric. Note that, in this case, we have $\sigma_2(0, 1) = 2\sigma_1(0, 1)$ by (6.1) so that (6.2) is not strict.

Even in the case $k = 1$, our proof differs a little bit from that of Fraser-Schoen [41]. And for higher eigenvalues, compared to the first, we have to deal with possible bubbling phenomena and thus to analyze precisely them in order to rule them out thanks to the gap assumption. The starting point of our proof is the following simple remark : it is somewhat more convenient (even if not easy) to maximize the Steklov eigenvalue among metrics in a given conformal class since everything depends then from a single function. Then we pick up a special maximizing sequence for $\sigma_k(\gamma, m)$ consisting in maximizers in their own conformal class. These maximizers come, as we shall see, with a corresponding harmonic map with free boundary from Σ into some Euclidean ball and the proof of Theorem 14 relies on a careful asymptotic analysis of these harmonic maps when the conformal class degenerates. Quantification results for such sequences of harmonic maps with free boundary were recently obtained in Laurain-Petrides [70].

In order to carry out this program, we introduce the conformal invariant

$$\sigma_k(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \sigma_k(\tilde{g}) L_{\tilde{g}}(\partial\Sigma)$$

for any smooth compact Riemannian surface (Σ, g) with a non-empty boundary. Here $[g]$ denotes the conformal class of g , that is all the metrics on Σ which are a multiple of g by a smooth positive function. Then, if Σ is orientable of genus γ and with m boundary components, we have of course that

$$\sigma_k(\gamma, m) = \sup_{[g]} \sigma_k(\Sigma, [g]) .$$

Once again, one can prove by standard gluing techniques (see [30]) that

$$\sigma_k(\Sigma, [g]) \geq \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} \left(\sigma_{k-j}(\Sigma, [g]) + \sum_{m=1}^s \sigma_{i_m}(\mathbb{D}, [\tilde{g}]) \right) . \quad (6.3)$$

Note that thanks to (6.1), this inequality reads completely as

$$\sigma_k(\Sigma, [g]) \geq \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} (\sigma_{k-j}(\Sigma, [g]) + 2\pi j)$$

but, for a reason which will become clear in the proofs, we prefer to state it in the form of (6.3). Then we have the following existence result :

Theorem 15. *Let (Σ, g) be a compact Riemannian surface with a non empty smooth boundary. Let $k \geq 1$. Then, if (6.3) is strict, there exists a smooth maximal metric $\tilde{g} \in [g]$, such that $\sigma_k(\Sigma, [g]) = \sigma_k(\Sigma, \tilde{g})L_{\tilde{g}}(\Sigma)$.*

Note that by (6.1) and (6.3), the gap condition of our theorem would be a consequence of

$$\sigma_k(\Sigma, [g]) > \sigma_{k-1}(\Sigma, [g]) + 2\pi .$$

If a maximal metric \tilde{g} for $\sigma_k(\Sigma, [g])$ exists, the conformal factor related to g of a maximal metric \tilde{g} , is $\Phi \cdot \partial_\nu \Phi$ on $\partial\Sigma$, where Φ is some harmonic map from Σ into \mathbb{B}^{n+1} with free boundary, map whose coordinates are eigenfunctions for the k -th Steklov eigenvalue. Such a map takes value in the Euclidean ball, is harmonic inside Σ , satisfies that $|\Phi| = 1$ and $\partial_\nu \Phi$ is orthogonal to $T_\Phi \mathbb{S}^n$ on the boundary of Σ . These harmonic maps with free boundary have been studied in particular in Da Lio [24], Da Lio-Rivière [25], Laurain-Petrides [70] and Scheven [102].

The strategy of proof of Theorem 15 is the following. We do not prove either that any maximizing sequence does converge, up to a subsequence, to a maximizer nor that maximizers in a possible "weaker sense" are regular. Instead, as was initiated by Fraser-Schoen [41], we carefully select a maximizing sequence by a regularization process which does converge to a smooth maximizer. This special maximizing sequence is the solution of an approached variational problem and comes with a sequence of "almost" harmonic maps with free boundary in some Euclidean ball. The core of the proof is to carefully analyze the asymptotic behaviour of these maps to prove that they do converge to a real smooth harmonic map with free boundary, leading to a maximal metric for the Steklov eigenvalue under consideration. The main difficulty is that, contrary to the case $k = 1$, one can not a priori avoid phenomenon of concentration, with multiple bubbles appearing. We thus have to perform a bubble tree decomposition for this sequence, to understand precisely the behaviour of these maps at a concentration point, to prove a no-neck energy result, in order to get a quantification result, and enough test-functions to use the variational characterization of the k -th Steklov eigenvalue in order to violate the gap assumption of the theorem.

The proof of Theorem 14 starts from the existence of maximal metrics in their own conformal class : this gives once again a special maximizing sequence. We then understand the behaviour of this sequence if the conformal class degenerates in order to prove that it can not happen under the gap assumption of the theorem. Then we rely on a compactness result by Laurain-Petrides [70] to finally prove that our maximizing sequence does converge to a smooth maximizer once degeneracy of the conformal class has been ruled out.

Analogous questions can be considered concerning the maximization of Laplace eigenvalues on closed surfaces. Inequalities (6.2) and (6.3) were proved in this situation by Colbois-El Soufi [21]. Maximizing metrics for Laplace eigenvalues come with minimal immersion of the surface into some sphere. If one adds the conformal class constraint, they come with smooth harmonic maps into the sphere. The analog of Theorem 14 for Laplace eigenvalues was proved in Petrides [95]. The analog of Theorem 15 was recently announced with a very brief sketch of proof in Nadirashvili-Sire [86] and proved in Petrides [95]. The proofs in the Steklov case are

somewhat more difficult since one has to deal with "almost" harmonic maps with free boundary in some Euclidian ball instead of "almost" harmonic maps in some sphere. The analysis of such maps is more tricky : regularity and quantification results are for instance more recent (see Scheven [102], Da Lio-Rivière [25], Da Lio [24], Laurain-Petrides [70] compared to Hélein [51], Parker [88]) and the description of the bubbling phenomenon in the case of the present paper was explicitly asked for by Fraser-Schoen [39].

The paper is organized as follows :

In Section 6.2, we introduce some notations and recall some more or less classical tools that we shall use during the proof. Section 6.3 is devoted to the set up of the proof of Theorem 15, proof carried out in Sections 6.4 to 6.6. We refer to the end of Section 6.3 for a detailed sketch of the proof of Theorem 15.

We prove Theorem 14 in Section 6.7, dealing with a maximizing sequence of metrics for $\sigma_k(\gamma, m)$ whose k -th eigenvalue is maximal in its conformal class. We then study the asymptotics of the harmonic maps on Σ with free boundary into some \mathbb{B}^{n+1} they define, and thanks to the gap assumption of the theorem, we remove all the problems of convergence which could occur for this maximizing sequence.

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6.2 Preliminaries

6.2.1 Notations

Let (M, g) be a smooth Riemannian surface with a boundary of length $L_g(\partial M) = 1$. Let $x \in M$ and $r > 0$. We denote by $B_g(x, r)$ the open ball of radius r centered at x . If $x \in \partial M$, we let $I_g(x, r) = \partial M \cap B_g(x, r)$. In the Euclidean upper half-plane $\mathbb{R}_+^2 = \{(s, t) \in \mathbb{R}^2; t \geq 0\}$, we let for $x \in \mathbb{R} \times \{0\}$, $\mathbb{D}_r^+(x) = \mathbb{D}_r(x) \cap \mathbb{R}_+^2$ and $I_r(x) = (-r, r) \times \{0\}$.

We denote by $\mathcal{M}(\partial M)$ the set of positive Radon measures equipped with the weak* topology on ∂M and by $\mathcal{M}_1(\partial M)$ the subset of probability measures.

As already said, we denote by $\sigma_k(M, g)$ the k -th eigenvalue of the Dirichlet-to-Neumann operator on M . It satisfies the classical min-max variational characterization :

$$\sigma_k(M, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_{\partial M} \phi^2 d\sigma_g}, \quad (6.4)$$

where the infimum is taken over the spaces of smooth functions E_{k+1} of dimension $k + 1$.

For an open set $\Omega \subset M$ such that $\partial\Omega = \Gamma \cup \tilde{\Gamma}$ where $\Gamma = \partial\Omega \cap \partial M$ and $\tilde{\Gamma} = \partial\Omega \setminus \partial M$ are non-empty piecewise smooth curves, and a smooth density e^u on Γ we denote by $\sigma_*(\Omega, g, \Gamma, e^u)$ the first eigenvalue for the following problem

$$\begin{cases} \Delta_g \phi = 0 & \text{in } \Omega \\ \partial_\nu \phi = \sigma_*(\Omega, g, \Gamma, e^u) e^u \phi & \text{on } \Gamma \\ \phi = 0 & \text{on } \tilde{\Gamma}, \end{cases}$$

that is

$$\sigma_*(\Omega, g, \Gamma, e^u) = \inf_{\phi \in H} \frac{\int_{\Omega} |\nabla \phi|_g^2 dv_g}{\int_{\Gamma} \phi^2 e^u d\sigma_g},$$

where

$$H = \{\phi \in W^{1,2}(\Omega), \phi = 0 \text{ on } \tilde{\Gamma}\},$$

the value of ϕ on $\partial\Omega$ being understood taken in the sense of the Sobolev trace.

For all the paper, we fix $\delta > 0$, a constant $C_0 > 1$ and a family $(x_l)_{l=1,\dots,L}$ of points in ∂M and smooth functions $v_l : M \mapsto \mathbb{R}$ such that

- for any $l \in \{1, \dots, L\}$, $g_l = e^{-2v_l} g$ is a flat metric in $\Omega_l = B_{g_l}(x_l, 2\delta)$, and $\Gamma_l = I_{g_l}(x_l, 2\delta)$ is a geodesic line for g_l so that the exponential map \exp_{g_l, x_l} defines an isometry between $\mathbb{D}_{2\delta}^+(0)$ and $(B_{g_l}(x_l, 2\delta), g_l)$
 - $\partial M = \bigcup_{l=1}^L \gamma_l$ where $\gamma_l = I_{g_l}(x_l, \delta)$.
 - For any $1 \leq l \leq L$, $C_0^{-2} \leq e^{2v_l} \leq C_0^2$.
 - For any $x \in \omega_l$ and $0 < r < \delta$, $B_g(x, C_0^{-1}r) \subset B_{g_l}(x, r) \subset B_g(x, C_0 r)$
- For $1 \leq l \leq L$ and a point $z \in \mathbb{D}_{2\delta}^+(0)$, we let

$$e^{2\tilde{v}_l(z)} = e^{2v_l(\exp_{g_l, x_l}(z))} \text{ and } \tilde{z}^l = \exp_{g_l, x_l}(z)$$

and for $x \in \Omega_l$ and a set $\Omega \subset \Omega_l$,

$$\tilde{x}^l = \exp_{g_l, x_l}^{-1}(x) \text{ and } \tilde{\Omega}^l = \exp_{g_l, x_l}^{-1}(\Omega).$$

For a smooth density e^u on ∂M we let

$$e^{\tilde{u}^l(z)} = e^{\tilde{v}_l(z)} e^{u(\exp_{g_l, x_l}(z))}$$

so that for $\Gamma \subset \Gamma_l$,

$$\int_{\Gamma} e^u d\sigma_g = \int_{\tilde{\Gamma}^l} e^{\tilde{u}^l} ds.$$

For other functions $\phi \in L^1(M)$ or measures $\nu \in \mathcal{M}(\partial M)$, we let

$$\tilde{\phi}^l(z) = \phi(\exp_{g_l, x_l}(z)) \text{ and } \tilde{\nu}^l = \exp_{g_l, x_l}^*(\nu).$$

Let $p_{\epsilon}(x, y)$ be the heat kernel of ∂M at time $\epsilon > 0$ for the induced measure $d\sigma_g$. Then, for $y, z \in \Gamma_l$, we let

$$\tilde{p}_{\epsilon}^l(z, y) = e^{\tilde{v}_l(z)} p_{\epsilon}(\exp_{g_l, x_l}(z), \exp_{g_l, x_l}(y))$$

so that for a density $e^{u(x)} = \int_{\Gamma} p_{\epsilon}(x, y) d\nu(y)$ for $\Gamma \subset \Gamma_l$ and some measure ν , we have

$$e^{\tilde{u}^l(z)} = \int_{\tilde{\Gamma}^l} \tilde{p}_{\epsilon}^l(z, y) d\tilde{\nu}(y)$$

and for $\phi \in L^1(\partial M)$,

$$\int_{\tilde{\Gamma}^l} \tilde{\phi}^l(s, 0) \tilde{p}_{\epsilon}^l((s, 0), \tilde{y}^l) ds = \int_{\Gamma} \phi(x) p_{\epsilon}(x, y) d\sigma_g(x).$$

When the context is clear, we drop the exponent l in all the notations.

Now, for parameters $a \in \mathbb{R} \times \{0\}$ and $\alpha > 0$, we define the following rescaled objects

$$\hat{x} = \frac{\tilde{x} - a}{\alpha}, \hat{\Omega} = \frac{\tilde{\Omega} - a}{\alpha}, \hat{\Gamma} = \frac{\tilde{\Gamma} - a}{\alpha},$$

$$e^{2\hat{u}(z)} = \alpha^2 e^{2\tilde{u}(\alpha z + a)}, \hat{\phi}(z) = \tilde{\phi}(\alpha z + a), \hat{v} = H_{a,\alpha}^*(\tilde{v}), \hat{p}_\epsilon(z, y) = \alpha \tilde{p}_\epsilon^l(\alpha z + a, \alpha y + a),$$

where $H_{a,\alpha}(x) = \alpha x + a$, so that if $e^{u(x)} = \int_{\Gamma} p_\epsilon(x, y) d\nu(y)$, we have

$$e^{\hat{u}(z)} = \int_{\hat{\Gamma}} \hat{p}_\epsilon(z, y) d\hat{\nu}(y)$$

and

$$\int_{\hat{\Gamma}} \phi((s, 0)) \hat{p}_\epsilon((s, 0), \hat{y}) ds = \int_{\Gamma} \phi(x) p_\epsilon(x, y) d\sigma_g(y).$$

We also denote for $z \in \mathbb{R}^2$,

$$\check{z} = \exp_{g_l, x_l}(\alpha z + a)$$

so that $\hat{z} = z$ and

$$\check{\Omega} = \exp_{g_l, x_l}(\alpha \Omega + a).$$

6.2.2 Estimates on the heat kernel

The heat kernel $p_\epsilon(x, y)$ of a the union of circles ∂M at time $\epsilon > 0$ with respect to the measure $d\sigma_g$ satisfies the following uniform estimates as $\epsilon \rightarrow 0$

$$p_\epsilon(x, y) =_{\epsilon \rightarrow 0} \frac{e^{-\frac{d_g(x, y)^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} (a_0(x, y) + \epsilon a_1(x, y) + \epsilon^2 a_2(x, y) + o(\epsilon^2)) \quad (6.5)$$

with $a_0, a_1, a_2 \in \mathcal{C}^\infty(\partial M \times \partial M)$ are Riemannian invariants such that $a_0(x, x) = 1$ as proved for instance in [6]. We have also a uniform bound : there exists $A_0 > 0$ such that for any $\epsilon > 0$,

$$\forall x, y \in \partial M, \frac{1}{A_0 \sqrt{4\pi\epsilon}} e^{-\frac{d_g(x, y)^2}{4\epsilon}} \leq p_\epsilon(x, y) \leq \frac{A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(x, y)^2}{4\epsilon}}. \quad (6.6)$$

We deduce the same uniform properties for the rescaled heat kernel $\hat{p}_\epsilon(x, y)$ by some parameters $a_\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_\epsilon > 0$ such that $a_\epsilon \rightarrow a \in \mathbb{R} \times \{0\}$ and $\alpha_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. We have for any $R > 0$,

$$\hat{p}_\epsilon(z, y) = \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+o(1))}}{\sqrt{4\pi\theta_\epsilon}} (1 + o(1)) \text{ uniformly on } \mathbb{D}_R \times \mathbb{D}_R, \quad (6.7)$$

where $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)} \alpha_\epsilon^2}$ and we have the following bound for any fixed $0 < \rho < 1$

$$\frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+\rho)}}{\sqrt{4\pi\theta_\epsilon}} (1 - \rho) \leq \hat{p}_\epsilon(z, y) \leq \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1-\rho)}}{\sqrt{4\pi\theta_\epsilon}} (1 + \rho) \quad (6.8)$$

for all $\epsilon > 0$ small enough.

Let's prove (6.7). We fix $R > 0$ and we have uniformly for $(x, y) \in I_R \times I_R$ as $\epsilon \rightarrow 0$

$$\begin{aligned}\hat{p}_\epsilon(x, y) &= \frac{\alpha_\epsilon e^{v_l(\check{x})}}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(\check{x}, \check{y})^2}{4\epsilon}} (a_0(\check{x}, \check{y}) + o(1)) \\ &= \frac{\alpha_\epsilon e^{\tilde{v}_l(a)}}{\sqrt{4\pi\epsilon}} (1 + o(1)) e^{-\frac{d_g(\check{x}, \check{y})^2}{4\epsilon}}\end{aligned}$$

by (6.5). It remains to notice that

$$d_g(\check{x}, \check{y}) = e^{\tilde{v}_l(a)} |x - y| \alpha_\epsilon (1 + o(1))$$

uniformly for $(x, y) \in \mathbb{D}_R^+ \times \mathbb{D}_R^+$ and we get the desired approximation (6.7).

For a sequence of measures $\nu_\epsilon \in \mathcal{M}(\partial M)$, we also have uniform bounds for $R > r > 0$ and $\theta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$

$$\sup_{x \in I_{R-r}} \int_{\partial M \setminus \check{I}_R} \alpha_\epsilon p_\epsilon(\check{x}, y) d\nu_\epsilon(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}}\right). \quad (6.9)$$

We prove it thanks to (6.6) and (6.8). Let $x \in I_{R-r}$.

$$\begin{aligned}\alpha_\epsilon \int_{\partial M \setminus \check{I}_R} p_\epsilon(\check{x}, y) d\nu_\epsilon(y) &= e^{-v_l(\check{x})} \int_{I_{C_0^2 R} \setminus I_R} \hat{p}_\epsilon(x, z) d\hat{\nu}_\epsilon(z) \\ &\quad + \int_{\partial M \setminus \check{I}_{C_0^2 R}} \alpha_\epsilon p_\epsilon(\check{x}, y) d\nu_\epsilon(y) \\ &\leq C_0 \int_{I_{C_0^2 R} \setminus I_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}} d\hat{\nu}_\epsilon(z) \\ &\quad + \int_{\partial M \setminus I_g(\bar{a}_\epsilon, \frac{\alpha_\epsilon C_0^2 R}{C_0})} \frac{\alpha_\epsilon A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(\check{x}, y)^2}{4\epsilon}} d\nu_\epsilon(y) \\ &\leq O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}}\right) + \frac{A_0 \alpha_\epsilon}{\sqrt{4\pi\epsilon}} e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}},\end{aligned}$$

where $\check{I}_r \subset I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r) \subset I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 R)$. This proves (6.9). We also have

$$\sup_{x \in \partial M \setminus \check{I}_R} \int_{\check{I}_r} p_\epsilon(x, y) d\sigma_g(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}}\right). \quad (6.10)$$

Let $x \in \partial M \setminus \check{I}_R$. We assume that $x \in I_{C_0^2 R} \setminus I_R$. Then,

$$\begin{aligned}\int_{\check{I}_r} p_\epsilon(x, y) d\sigma_g(y) &= \int_{\check{I}_r} \hat{p}_\epsilon(z, \check{x}) dz \\ &\leq \frac{1}{\sqrt{\pi\theta_\epsilon}} \int_{\check{I}_r} e^{-\frac{|x-z|^2}{8\theta_\epsilon}} dz \\ &\leq \frac{2r}{\sqrt{\pi\theta_\epsilon}} e^{-\frac{(R-r)^2}{8\theta_\epsilon}}\end{aligned}$$

Now, if ϵ is small enough and if $x \in \partial M \setminus \check{I}_{C_0^2 R} \subset \partial M \setminus I_g(\bar{a}_\epsilon, \alpha_\epsilon R C_0)$, we have

$$\begin{aligned} \int_{I_r} p_\epsilon(x, y) d\sigma_g(y) &\leq \int_{I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} p_\epsilon(x, y) d\sigma_g(y) \\ &\leq \frac{A_0}{\sqrt{4\pi\epsilon}} \int_{I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} e^{-\frac{d_g(x, y)^2}{4\epsilon}} d\sigma_g(y) \\ &\leq O\left(\frac{e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}}}{\sqrt{\theta_\epsilon}}\right). \end{aligned}$$

We proved (6.10). Now let's prove that

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \left| \int_{I_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0 \quad (6.11)$$

We fix $0 < \rho < \frac{1}{2}$ and $R > 0$. Then, for ϵ small enough, we have by (6.8) that

$$\int_{I_R} \hat{p}_\epsilon(z, x) dz \leq \int_{\mathbb{R} \times \{0\}} \frac{e^{-\frac{|x-z|^2(1-\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}} (1+\rho) dz = \frac{1+\rho}{\sqrt{1-\rho}}$$

for any $x \in I_r$ and

$$\begin{aligned} \int_{I_R} \hat{p}_\epsilon(z, x) dz &\geq \int_{I_R} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}} (1-\rho) dz \\ &\geq \int_{\mathbb{R} \times \{0\}} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}} (1-\rho) dz - \int_{\mathbb{R} \times \{0\} \setminus I_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\epsilon}} dz \\ &\geq \frac{1-\rho}{\sqrt{1+\rho}} + o(1) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

uniformly on I_r . Letting $\epsilon \rightarrow 0$, then $R \rightarrow +\infty$ and then $\rho \rightarrow 0$ gives (6.11).

6.2.3 Capacity and Poincaré inequalities

We first notice the following consequence of the classical computation of the capacity of annuli in \mathbb{R}^2 .

Claim 40. *Let (M, g) be a compact Riemannian surface. Then, there is $C > 0$ and $r_0 > 0$ such that for all $x \in M$ and all $0 < r_2 < r_1 < r_0$, there exists a smooth function $\eta_{g, x, r_1, r_2} : M \rightarrow \mathbb{R}$ with*

- $0 \leq \eta_{g, x, r_1, r_2} \leq 1$
- $\eta_{g, x, r_1, r_2} = 1$ on $B_g(x, r_2)$
- $\eta_{g, x, r_1, r_2} \in C_c^\infty(B_g(x, r_1))$
- $\int_M |\nabla \eta_{g, x, r_1, r_2}|_g^2 dv_g \leq \frac{C}{\ln\left(\frac{r_1}{r_2}\right)}.$

We now recall two theorems giving Poincaré inequalities on surfaces.

Theorem 16 ([1], Lemma 8.3.1). *Let (M, g) be a Riemannian manifold. Then, there exists a constant $B > 0$ such that for any $L \in W^{-1,2}(M)$ with $L(1) = 1$, we have the following Poincaré inequality*

$$\forall f \in W^{1,2}(M), \int_M (f - L(f))^2 dv_g \leq B \|L\|_{W^{-1,2}(M)}^2 \int_M |\nabla f|_g^2 dv_g .$$

We denote by

$$C_{1,2}(K) = \inf \left\{ \int_{\mathbb{R}^2} \phi^2 dv_g + \int_{\mathbb{R}^2} |\nabla \phi|_g^2 dv_g; \phi \in C_c^\infty(\mathbb{R}^2), \phi \geq 1 \text{ on } K \right\}$$

the capacity of a compact set $K \subset \mathbb{R}^2$ and

$$Cap_2(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \phi|_g^2 dv_g; \phi \in C_c^\infty(\Omega), \phi \geq 1 \text{ on } K \right\}$$

the relative capacity of $K \subset \subset \Omega$.

Theorem 17 ([1], Corollary 8.2.2). *Let $\Omega \subset \mathbb{R}^2$ be a bounded extension domain. Then, there exists a constant C_Ω such that for any compact $K \subset \Omega$ with $C_{1,2}(K) > 0$ and for any function $f \in C^\infty(\Omega)$ such that $f = 0$ on K ,*

$$\|f\|_{L^2(\Omega)} \leq \frac{C_\Omega}{C_{1,2}(K)} \|\nabla f\|_{L^2(\Omega)} .$$

Ω is a bounded extension domain means that the extension by 0 on \mathbb{R}^2 of every function in $W_0^{1,2}(\Omega)$ is $W^{1,2}$ in \mathbb{R}^2 . This is true for the family of sets we consider during the proof :

$$\Omega = \mathbb{D}_{\frac{1}{\rho}}^+ \setminus \bigcup_{i=1}^s \mathbb{D}_\rho(x_i) ,$$

where $\rho > 0$, $x_i \in \mathbb{D}_{\frac{1}{\rho}}$ such that if $i \neq j$, then $x_i \neq x_j$ and

$$10\rho < \min \left(\min_i d(x_i, \partial \mathbb{D}_{\frac{1}{10\rho}}); \min_{i \neq j} \frac{|x_i - x_j|}{2} \right) .$$

We now set

$$\Omega_K = \mathbb{D}_{\frac{1}{K\rho}}^+ \setminus \bigcup_{i=1}^s \mathbb{D}_{K\rho}(x_i)$$

for some fixed number $1 < K < 10$ chosen independent of the problem we consider. We obtain the corollary :

Corollary. *Let $r > 0$ fixed. Then, we have a constant $C_r > 0$ such that for every $f \in C^\infty(\Omega)$ which vanishes on a smooth piecewise curve $\Gamma \subset \subset \Omega_K$ which connects two points of distance $r > 0$,*

$$\|f\|_{L^2(\Omega)} \leq C_r \|\nabla f\|_{L^2(\Omega)} .$$

Indeed, it is proved in ([52], pages 95-97) that

$$Cap_2(\Gamma, \Omega) \geq \frac{K_0}{\ln(\frac{1}{r})}$$

and that

$$C_{1,2}(\Gamma) \geq K_1 Cap_2(\Gamma, \Omega)$$

for constants $K_0 > 0$ and $K_1 > 0$ which only depend on Ω and K .

6.3 Selection of a maximizing sequence

We fix $k \geq 1$. In this section, we build a specific maximizing sequence for $\sigma_k(M, [g])$ thanks to the heat equation on ∂M . Let $\epsilon > 0$. We denote by K_ϵ the heat operator on ∂M so that for a positive Radon measure $\nu \in \mathcal{M}(M)$, $K_\epsilon[\nu]d\sigma_g$ is the solution at time $\epsilon > 0$ of the heat equation on the curves $(\partial M, d\sigma_g)$ which converges to ν as $\epsilon \rightarrow 0$ in $\mathcal{M}(\partial M)$. Given $x, y \in \mathcal{M}(\partial M)$, we denote by $p_\epsilon(x, y)$ the heat kernel of $(\partial M, g)$ so that for $\nu \in \mathcal{M}(\partial M)$,

$$K_\epsilon[\nu](x) = \int_{\partial M} p_\epsilon(x, y) d\nu(y) .$$

For $f \in L^1(\partial M)$, we set $K_\epsilon[f] := K_\epsilon[f d\sigma_g]$ so that

$$\int_{\partial M} K_\epsilon[f] d\nu = \int_{\partial M} f K_\epsilon[\nu] d\sigma_g .$$

For $\epsilon > 0$, we set

$$\sigma_\epsilon = \sup_{\nu \in \mathcal{M}(\partial M)} \sigma_k(M, g, \partial M, K_\epsilon[\nu]) . \quad (6.12)$$

By continuity of $\nu \in \mathcal{M}_1(\partial M) \mapsto \sigma_k(M, g, \partial M, K_\epsilon[\nu])$, a maximizing sequence for the variational problem (6.12) converges in $\mathcal{M}_1(\partial M)$, up to the extraction of a subsequence, to a measure $\nu_\epsilon \in \mathcal{M}_1(\partial M)$ such that

$$\sigma_\epsilon = \sigma_k(M, g, \partial M, K_\epsilon[\nu_\epsilon]) . \quad (6.13)$$

We set

$$e^{\mu_\epsilon} = K_\epsilon[\nu_\epsilon] \quad (6.14)$$

a sequence of smooth positive densities satisfying

$$\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{\mu_\epsilon}) \rightarrow \sigma_k(M, [g]) \text{ as } \epsilon \rightarrow 0 . \quad (6.15)$$

Indeed, $\sigma_\epsilon \leq \sigma_k(M, [g])$ for all $\epsilon > 0$ and for $\eta > 0$, there exists some density e^μ such that $\int_{\partial M} e^\mu d\sigma_g = 1$ and $\sigma_k(M, g, \partial M, e^\mu) \geq \sigma_k(M, [g]) - \frac{\eta}{2}$. By uniform estimates on the heat operator, $K_\epsilon[e^\mu] \rightarrow e^\mu$ as $\epsilon \rightarrow 0$ in $C^0(\partial M)$. Then, there exists $\epsilon_0 > 0$ such that

$$\sigma_\epsilon \geq \sigma_k(M, g, \partial M, K_\epsilon[e^\mu]) \geq \sigma_k(M, g, \partial M, e^\mu) - \frac{\eta}{2} \geq \sigma_k(M, [g]) - \eta$$

for $\epsilon < \epsilon_0$. We get (6.15). Now, thanks to the choice of the maximizing sequence (6.14) the variational problem (6.12) gives

Proposition 6. Fix $\epsilon > 0$. Then, there exists a family $\Phi_\epsilon = (\phi_\epsilon^0, \dots, \phi_\epsilon^{n(\epsilon)})$ of smooth independent functions in $L^2(\partial M, e^{\mu_\epsilon} d\sigma_g)$ such that

(i) For $i \in \{0, \dots, n(\epsilon)\}$, $\phi_\epsilon^i \in E_k(M, g, \partial M, e^{\mu_\epsilon})$, that is to say

$$\begin{cases} \Delta_g \phi_\epsilon^i = 0 & \text{in } M \\ \partial_\nu \phi_\epsilon^i = \sigma_\epsilon e^{\mu_\epsilon} \phi_\epsilon^i & \text{in } \partial M \end{cases}$$

(ii) $K_\epsilon[|\Phi_\epsilon|^2] \geq 1$ on ∂M ,

(iii) $K_\epsilon[|\Phi_\epsilon|^2] = 1$ on $\text{supp}(\nu_\epsilon)$.

Proof. Since ϵ is fixed, we omit the indices ϵ for σ_ϵ , ν_ϵ and e^{u_ϵ} up to the end of the proof of the claim.

Let $\mu \in \mathcal{M}(\partial M)$ and $t > 0$. We set $\sigma_t = \sigma_k(M, g, \partial M, K_\epsilon[\nu + t\mu])$. Note that $\sigma = \sigma_{t=0}$ and by continuity, $\sigma_t \rightarrow \sigma$ as $t \rightarrow 0^+$. We first prove that

$$\lim_{t \rightarrow 0^+} \frac{\sigma_t - \sigma}{t} = \inf_{\phi \in E_k(M, g, \partial M, e^u)} \left(-\sigma \frac{\int_{\partial M} K_\epsilon[\phi^2] d\mu}{\int_{\partial M} \phi^2 e^u d\sigma_g} \right) \quad (6.16)$$

Let $\phi_0, \phi_1, \dots, \phi_k$ an orthonormal family in $L^2(\partial M, e^u d\sigma_g)$ such that $\phi_i \in E_i(M, g, \partial M, e^u)$. We set $E = \text{Vect}\{\phi_0, \dots, \phi_k\}$. Then, by the min-max variational characterization (6.4),

$$\begin{aligned} \sigma_t &\leq \sup_{\phi \in E \setminus \{0\}} \left(\frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_{\partial M} \phi^2 K_\epsilon[\nu + t\mu] d\sigma_g} \right) \\ &= \sup_{\phi \in E \cap S^k} \left(\frac{\int_M |\nabla \phi|_g^2 dv_g}{\int_{\partial M} \phi^2 K_\epsilon[\nu] d\sigma_g + t \int_{\partial M} K_\epsilon[\phi^2] d\mu} \right) \end{aligned}$$

where $S^k = \{\sum_{i=0}^k \beta_i \phi_i, \beta \in \mathbb{S}^k\}$ and

$$\begin{aligned} \sigma_t &\leq \sup_{\phi = \sum_{i=0}^k \beta_i \phi_i \in S^k} \left(\sum_{i=0}^k \beta_i^2 \sigma_i(M, g, \partial M, e^u) \left(1 - t \int_{\partial M} K_\epsilon[\phi^2] d\mu + o(t) \right) \right) \\ &\leq \sigma \left(1 - t \frac{\int_{\partial M} K_\epsilon[\phi_k^2] d\mu}{\int_{\partial M} \phi_k^2 e^u d\sigma_g} + o(t) \right) \end{aligned}$$

uniformly as $t \rightarrow 0$. Indeed, $\sigma = \sigma_k(M, g, \partial M, e^u) > \sigma_{k-1}(M, g, \partial M, e^u)$ by the gap $\sigma_k(M, [g]) \geq \sigma_{k-1}(M, [g]) + 2\pi$ and since we have (6.15). Then, minimizing among the $\phi_k \in E_k(M, g, \partial M, e^u)$, we get that

$$\limsup_{t \rightarrow 0^+} \frac{\sigma_t - \sigma}{t} \leq \inf_{\phi \in E_k(M, g, \partial M, e^u)} \left(-\sigma \frac{\int_{\partial M} K_\epsilon[\phi^2] d\mu}{\int_{\partial M} \phi^2 e^u d\sigma_g} \right) \quad (6.17)$$

Now, we let $\phi_t \in E_k(M, g, \partial M, K_\epsilon[\nu + t\mu])$ with $\|\phi_t\|_{L^2(\partial M, K_\epsilon[\nu] d\sigma_g)} = 1$. We have that

$$\begin{cases} \Delta_g \phi_t = 0 & \text{in } M \\ \partial_\nu \phi_t = \sigma_t K_\epsilon[\nu + t\mu] \phi_t = \sigma_t (e^u + t K_\epsilon[\mu]) \phi_t & \text{in } \partial M \end{cases} \quad (6.18)$$

For $t \leq \frac{\|e^u\|_{L^\infty}}{2\|K_\epsilon[\mu]\|_{L^\infty}}$, we have that

$$\frac{1}{2} e^u \leq K_\epsilon[\nu + t\mu] \leq 2e^u$$

and that for any $\phi \in C^\infty(\partial M)$,

$$\frac{1}{2} \int_{\partial M} e^u \phi^2 \leq \int_{\partial M} \phi^2 K_\epsilon[\nu + t\mu] \leq 2 \int_{\partial M} \phi^2 e^u$$

so that $L^2(K_\epsilon[\nu + t\mu] d\sigma_g)$ and $L^2(K_\epsilon[\nu] d\sigma_g) = L^2(e^u d\sigma_g)$ define the same sets with equivalent norms and constants in the equivalence independent of t . Then, $\{\phi_t\}$ is bounded in $L^2(e^u d\sigma_g)$.

By elliptic regularity theory for the Dirichlet-to-Neumann operator with the equation (6.18), (see [109], Chapter 7.11, page 37), there exists $\phi \in E_k(M, g, \partial M, e^u)$ such that up to the extraction of a subsequence, $\phi_t \rightarrow \phi$ in $\mathcal{C}^m(M)$ as $t \rightarrow 0^+$ and $\|\phi\|_{L^2(\partial M, e^u d\sigma_g)} = 1$. We denote by Π the orthogonal projection on $E_k(M, g, \partial M, e^u)$ with respect to the $L^2(\partial M, e^u d\sigma_g)$ -norm. Then, we write (6.18) as

$$\begin{cases} \Delta_g \left(\frac{\phi_t - \Pi\phi_t}{\alpha_t} \right) = 0 & \text{in } M \\ \partial_\nu \left(\frac{\phi_t - \Pi\phi_t}{\alpha_t} \right) - \sigma_t e^u \left(\frac{\phi_t - \Pi\phi_t}{\alpha_t} \right) = \frac{\sigma_t - \sigma}{\alpha_t} e^u \phi_t + \frac{t}{\alpha_t} \sigma_t K_\epsilon[\mu] \phi_t & \text{in } \partial M \end{cases} \quad (6.19)$$

with

$$\alpha_t = \|\phi_t - \Pi\phi_t\|_{L^\infty} + t + (\sigma - \sigma_t). \quad (6.20)$$

Up to the extraction of a subsequence, we have that

$$t_0 = \lim_{t \rightarrow 0^+} \frac{t}{\alpha_t} \text{ and } \delta_0 = \lim_{t \rightarrow 0^+} \frac{\sigma - \sigma_t}{\alpha_t}.$$

Notice that $\delta_0 \geq 0$. By elliptic theory on the Dirichlet-to-Neumann operator (see [109], Chapter 7.11, page 37), since $\frac{\phi_t - \Pi\phi_t}{\alpha_t}$ is uniformly bounded as $t \rightarrow 0^+$, we get up to the extraction of a subsequence that

$$\frac{\phi_t - \Pi\phi_t}{\alpha_t} \rightarrow R_0 \text{ as } t \rightarrow 0^+ \text{ in } \mathcal{C}^m(M)$$

where $R_0 \in E_k(M, g, \partial M, e^u)^\perp$. Passing to the limit in the equation (6.19), we get

$$\begin{cases} \Delta_g R_0 = 0 & \text{in } M \\ \partial_\nu R_0 - \sigma e^u R_0 = -\delta_0 e^u \phi + t_0 \sigma K_\epsilon[\mu] \phi & \text{in } \partial M \end{cases} \quad (6.21)$$

and by (6.20)

$$\|R_0\|_\infty + t_0 + \delta_0 = 1. \quad (6.22)$$

Testing (6.21) against ϕ , and using the fact that $R_0 \in E_k(M, g, \partial M, e^u)^\perp$, we have that

$$\delta_0 = \delta_0 \int_{\partial M} e^u \phi^2 d\sigma_g = t_0 \sigma \int_{\partial M} K_\epsilon[\mu] \phi^2 d\sigma_g.$$

If $t_0 = 0$, then $\delta_0 = 0$ and then $R_0 = 0$ thanks to (6.21) and the fact that $R_0 \in E_k(M, g, \partial M, e^u)^\perp$. This is absurd with (6.22). Thus $t_0 \neq 0$ and

$$\lim_{t \rightarrow 0^+} \frac{\sigma_t - \sigma}{t} = \frac{-\delta_0}{t_0} = -\sigma \frac{\int_{\partial M} K_\epsilon[\phi^2] d\mu}{\int_{\partial M} \phi^2 e^u d\sigma_g}$$

This and (6.17) gives (6.16).

Since $(1 + t \int_{\partial M} d\mu) \sigma_t \leq \sigma$ for all $t \geq 0$, we deduce from (6.16) that

$$\forall \mu \in \partial \mathcal{M}, \exists \phi \in E_k(M, g, \partial M, e^u), \int_{\partial M} \phi^2 e^u d\sigma_g = 1 \text{ and } \int_{\partial M} (1 - K_\epsilon[\phi^2]) d\mu \leq 0. \quad (6.23)$$

We define the following subsets of $\mathcal{C}^0(\partial M)$

$$K = \{ \psi \in \mathcal{C}^0(M); \exists \phi_0, \dots, \phi_n \in E_k(M, g, \partial M, e^u), \psi = \sum_{i=0}^n K_\epsilon[\phi_i^2] - 1, \int_{\partial M} \psi d\nu = 0 \}$$

and

$$F = \{f \in \mathcal{C}^0(\partial M), f \geq 0\}.$$

F is closed and convex. The set K is convex since it is a translation of the convex hull of

$$C = \{K_\epsilon[\phi^2]; \phi \in E_k(M, g, \partial M, e^u), \|\phi\|_{L^2(M, g, \partial M, e^u)} = 1\}.$$

Since $E_k(M, g, \partial M, e^u)$ is finite dimensional, the vector space spanned by C is finite dimensional and C is compact. Caratheodory's theorem gives that K is compact.

If $F \cap K = \emptyset$, Hahn-Banach theorem gives the existence of some $\mu \in \mathcal{M}(\partial M)$ such that

$$\forall f \in F, \int_{\partial M} f d\mu \geq 0 \quad (6.24)$$

and

$$\forall \psi \in K, \int_{\partial M} \psi d\mu < 0. \quad (6.25)$$

Then, μ is a non zero, by (6.24), positive, by (6.25), measure and μ contradicts (6.23) by (6.25). Thus $F \cap K \neq \emptyset$ and there exists $\phi^0, \dots, \phi^n \in E_k(M, g, \partial M, e^u)$ with

$$\int_{\partial M} |\Phi|^2 e^u d\sigma_g = 1 \text{ and } K_\epsilon[|\Phi|^2] \geq 1, \quad (6.26)$$

where $\Phi = (\phi^0, \dots, \phi^n)$. By Gaussian decomposition of some non-negative quadratic form, we can assume that (ϕ^0, \dots, ϕ^n) is a family of independent eigenfunctions in $L^2(\partial M, e^u d\sigma_g)$ and satisfies (6.26). This gives (i) and (ii). We can write that

$$1 = \int_{\partial M} |\Phi|^2 e^u d\sigma_g = \int_{\partial M} K_\epsilon[|\Phi|^2] d\nu \geq \int_{\partial M} d\nu = 1.$$

Therefore, $K_\epsilon[|\Phi|^2] = 1$ ν -a.e and since $K_\epsilon[|\Phi|^2]$ is continuous, $K_\epsilon[|\Phi|^2] = 1$ on $\text{supp}(\nu)$. This gives (iii) and ends the proof of the claim. \diamond

By a result of Fraser-Schoen [39] and Karpukhin-Kokarev-Polterovich [62], there exists a bound for the multiplicity of k -th Steklov eigenvalues on surfaces which only depends on k and the topology of the surface. Therefore, up to the extraction of a subsequence, we assume in the following that $n(\epsilon) = n$ is fixed.

We organize the proof of Theorem 15 as follows :

In section 6.4, we give regularity estimates on the densities e^{u_ϵ} and on the associated Steklov eigenfunctions defined by Proposition 6 (see Claim 43). These estimates permit to pass to the limit on the eigenvalue equation (Proposition 6 (i)) as $\epsilon \rightarrow 0$ (see Claim 44). However, we cannot pass to the limit on the whole surface. We have to avoid some singularities for the maximizing sequence which could occur. We cannot remove a priori some concentration points of $\{e^{2u_\epsilon} dv_g\}$ even with the assumption that (6.3) is strict. Other harmless singularities are also carefully avoided (see Claim 42).

From Sections 6.5 to 6.7, we assume the existence of concentration points for the maximizing sequence and we aim at deducing the case of equality in (6.3). In Section 6.5, we detect all the concentration scales thanks to the construction of a bubble tree. This leads to the proof of Proposition 7, page 209.

We then give in Section 6.6 regularity estimates on the eigenfunctions at each scale of concentration and pass to the limit in the equation they satisfy. Notice that this work is divided into two subsections, depending on the speed of convergence to zero of the concentration scale α_ϵ as $\epsilon \rightarrow 0$.

Finally, in Section 6.7.1, capitalizing on the energy estimates for the limiting measures and equations given in Section 6.4.2 on M (see (6.54)), at the end of Section 6.6.1 (see (6.102)) and Section 6.6.2 (see (6.107)) on some discs \mathbb{D} , we both prove the regularity of the limiting measures at all the scales of concentration, and that no energy is lost in the necks in the bubbling process. This is given by Proposition 8, page 244. Thanks to this proposition, we prove in Section 6.7.2 that the presence of concentration points imply the case of equality in (6.3) by a suitable choice of test functions for the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$.

Therefore, since the specific maximizing sequence $\{e^{u_\epsilon} d\sigma_g\}$ does not concentrate with the assumption that (6.3) is strict, the end of the proof of Theorem 15 just uses the second part of Proposition 8 in Section 6.7.1.

6.4 Regularity estimates in the surface

6.4.1 Regularity estimates far from singularities

In this subsection, we aim at getting finer and finer regularity estimates on the eigenfunctions which appear in Proposition 6 and pass to the limit on the equation they satisfy. We denote by ν the weak* limit of ν_ϵ . Notice that ν is also the weak* limit of $\{e^{u_\epsilon} d\sigma_g\}$. Indeed, if $\zeta \in \mathcal{C}^0(\partial M)$,

$$\begin{aligned} \left| \int_{\partial M} \zeta (e^{u_\epsilon} d\sigma_g - d\nu_\epsilon) \right| &= \left| \int_{\partial M} (K_\epsilon[\zeta] - \zeta) d\nu_\epsilon \right| \\ &\leq \sup_M |K_\epsilon[\zeta] - \zeta| \end{aligned}$$

which goes to 0 as $\epsilon \rightarrow 0$ by uniform continuity of ζ .

Hypothesis (iii) in Proposition 6 gives uniform estimates on the eigenfunctions $\{\phi_\epsilon^i\}$ on sets of points which lie at a distance to $\text{supp}(\nu_\epsilon)$ asymptotically smaller than $\sqrt{\epsilon}$.

Claim 41. *For any $R > 0$ there exists a constant $C_R > 0$ such that for any sequence (x_ϵ) of points in ∂M , with $d_g(x_\epsilon, \text{supp}(\nu_\epsilon)) \leq R\sqrt{\epsilon}$, we have*

$$|\phi_\epsilon^i(x_\epsilon)| \leq C_R \text{ for all } \epsilon > 0$$

Proof. We refer the reader to Section 6.2.1 for the notations used during this proof. We can assume that $x_\epsilon \in \omega_l$ for $1 \leq l \leq L$ fixed and we set

$$\hat{\Phi}_\epsilon(x) = \tilde{\Phi}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)$$

for $x \in \mathbb{D}_{\delta\epsilon^{-\frac{1}{2}}} \cap \mathbb{R}_+^2$. Then,

$$\begin{cases} \Delta_{\tilde{\zeta}} \hat{\Phi}_\epsilon^i = 0 & \text{in } \mathbb{D}_{\delta\epsilon^{-\frac{1}{2}}}^+ \\ \partial_t \hat{\Phi}_\epsilon^i = -\sigma_\epsilon \sqrt{\epsilon} e^{\tilde{u}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)} \hat{\phi}_\epsilon^i & \text{in } I_{\delta\epsilon^{-\frac{1}{2}}} \end{cases}$$

for $0 \leq i \leq n$. By estimate (6.6) of Section 6.2.2, $\{\sqrt{\epsilon}p_\epsilon\}$ is uniformly bounded so that $\{\sqrt{\epsilon}e^{\tilde{u}_\epsilon^l(\sqrt{\epsilon}x+\tilde{x}_\epsilon^l)}\}$ is uniformly bounded. Now, we let $y_\epsilon \in \text{supp}(v_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) \leq R\sqrt{\epsilon}$. Thanks to Proposition 6, we have that $K_\epsilon[|\Phi_\epsilon|^2](y_\epsilon) = 1$. Let us write then with (6.6), Section 6.2.2 that for $\rho > 0$,

$$\begin{aligned} 1 = K_\epsilon[|\Phi_\epsilon|^2](y_\epsilon) &\geq \sum_{i=0}^n K_\epsilon\left[|\phi_\epsilon^i|^2\right](y_\epsilon) \\ &= \sum_{i=0}^n \int_{\partial M} p_\epsilon(y, y_\epsilon) \left(\phi_\epsilon^i(y)\right)^2 d\sigma_g(y) \\ &\geq \sum_{i=0}^n \frac{1}{A_0\sqrt{4\pi\epsilon}} e^{-\rho^2 C_0^2} \int_{I_g(y_\epsilon, 2\rho C_0\sqrt{\epsilon})} \left(\phi_\epsilon^i(y)\right)^2 d\sigma_g(y) \\ &\geq \sum_{i=0}^n \frac{1}{A_0\sqrt{4\pi C_0}} e^{-\rho^2 C_0^2} \int_{I_{2\rho}(\hat{z}_\epsilon)} \left(\hat{\phi}_\epsilon^i(z)\right)^2 dz \end{aligned}$$

where we set $\hat{z}_\epsilon = \frac{1}{\sqrt{\epsilon}}(\tilde{y}_\epsilon^l - \tilde{x}_\epsilon^l)$ so that, up to the extraction of a subsequence $\hat{z}_\epsilon \rightarrow z_0 \in \partial M$ as $\epsilon \rightarrow 0$ and we deduce from the previous inequality that, for any $\rho > 0$, $\hat{\phi}_\epsilon^i$ is bounded in $L^2(I_\rho(z_0))$. Thus, by elliptic regularity of the Dirichlet-to-Neumann operator (see Taylor [109], Chapter 7.11, page 37), we get that $\{\hat{\phi}_\epsilon^i\}$ is uniformly bounded in I_ρ by some constant D_ρ . Setting $C_R = D_{2C_0R}$ gives the claim. \diamond

Now, we will restrict the estimates on the eigenfunctions ϕ_ϵ^i far from some singularities which could appear.

A_{r,ε} : We say that a point $x \in \partial M$ satisfies **A_{r,ε}** for some $r > 0$ and some $\epsilon > 0$ if

$$\sigma_*(B_g(x, r), g, I_g(x, r), e^{u_\epsilon}) \leq \frac{\sigma_k(M, [g])}{2}$$

B_{r,ε} : We say that a point $x \in M$ satisfies **B_{r,ε}** for $r > 0$ and $\epsilon > 0$ if there exists $f \in E_k(M, g, \partial M, \{e^{u_\epsilon}\})$ such that $f(x) = 0$ and the Nodal set of f which contains x does not intersect $\partial B_g(x, r) \setminus \partial M$.

Note that if $r_1 < r_2$, **A_{r₁,ε}** \Rightarrow **A_{r₂,ε}** and **B_{r₁,ε}** \Rightarrow **B_{r₂,ε}**. We say that a point $x \in M$ satisfies **P_{r,ε}** for $r > 0$ and $\epsilon > 0$ if $x \in \partial M$ and x satisfies **A_{r,ε}** or if x satisfies **B_{r,ε}**. For a surface (M, g) , a sequence of densities $\{e^{u_\epsilon}\}$ on ∂M and $r > 0$, we define the singular set

$$X_r(M, g, \partial M, \{e^{u_\epsilon}\}) = \{x \in \Omega, \text{ there exists } \epsilon > 0 \text{ such that } x \text{ satisfies } \mathbf{P}_{r,\epsilon}\}.$$

Note that if $r_1 < r_2$, then $X_{r_1}(M, g, \partial M, \{e^{u_\epsilon}\}) \subset X_{r_2}(M, g, \partial M, \{e^{u_\epsilon}\})$. The following claim holds true

Claim 42. *There exists a sequence $\{e^{u_{\epsilon_m}}\}$ with $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$ and there exist some points $p_1, \dots, p_s \in \partial M$ with $0 \leq s \leq k$ such that*

$$— \forall \rho > 0, \exists r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \subset \bigcup_{i=1}^s B_g(p_i, \rho),$$

— For any subsequence $\{e^{u_{\epsilon_m(j)}}\}_{j \geq 0}$ of $\{e^{u_{\epsilon_m}}\}_{m \geq 0}$,

$$\forall \rho > 0, \forall r > 0, \forall 1 \leq i \leq s, X_r(M, g, \partial M, \{e^{u_{\epsilon_m(j)}}\}) \cap B_g(p_i, \rho) \neq \emptyset. \quad (6.27)$$

Proof. Assume by contradiction that for any sequence $\epsilon_m \rightarrow 0$, as $m \rightarrow +\infty$, for any series of s points $p_1, \dots, p_s \in \partial M$ with $0 \leq s \leq k$, there is $\rho > 0$ such that

$$\forall r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho) \neq \emptyset. \quad (6.28)$$

Thanks to this hypothesis, we will deduce by induction the following property \mathbf{H}_s for $1 \leq s \leq k+1$

\mathbf{H}_s : There exist sequences $\epsilon_m \rightarrow 0$, $r_m \searrow 0$ as $m \rightarrow +\infty$, some points $p_1^m, \dots, p_s^m \in M$ and s pairwise distinct points $p_1, \dots, p_s \in \partial M$ such that for $1 \leq i \leq s$, $p_i^m \rightarrow p_i$ as $m \rightarrow +\infty$ and p_i^m satisfies $\mathbf{P}_{r_m, \epsilon_m}$.

Let's first prove \mathbf{H}_1 . By (6.28) applied for $s = 0$ and a sequence $\{2^{-j}\}$, we have for any fixed $m \geq 0$, the existence of $p_1^m \in X_{2^{-m}}(M, g, \partial M, \{e^{u_{2^{-j}}}\}_{j \geq 0})$. For $m \geq 0$, we choose $\epsilon_m = 2^{-j(m)}$ such that p_1^m satisfies $\mathbf{P}_{2^{-m}, \epsilon_m}$. It is clear that $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$. Up to the extraction of a subsequence, there exists $p_1 \in M$ such that $p_1^m \rightarrow p_1$ as $m \rightarrow +\infty$. Now, it is clear that $p_1 \in \partial M$. Indeed, if $p_1 \in M \setminus \partial M$, then we choose $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $B_g(p_1^m, r_m) \subset M \setminus \partial M$. Then p_1^m satisfies $\mathbf{B}_{r_m, \epsilon_m}$ and the Nodal set of some function $f_m \in E_k(M, g, \partial M, \{e^{u_{\epsilon_m}}\})$ which contains p_1^m does not intersect ∂M since it does not intersect $\partial B_g(p_1^m, r_m)$. Since f_m is harmonic, it vanishes on an open set of M by the maximum principle so that f_m vanishes on M . This contradicts the fact that f_m is a k -th eigenfunction for the Dirichlet-to-Neumann operator. Then $p_1 \in \partial M$ and we get \mathbf{H}_1 .

We assume now that \mathbf{H}_s is true for some $1 \leq s \leq k$. We consider the sequences $\{\epsilon_m\}$, $\{r_m\}$, $\{p_i^m\}$ and $p_1, \dots, p_s \in \partial M$ given by \mathbf{H}_s . Let us prove \mathbf{H}_{s+1} . By (6.28), there is $\rho > 0$ such that for all $r > 0$,

$$X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho) \neq \emptyset.$$

Let $p_{s+1}^m \in X_{r_m}(M, g, \partial M, \{e^{u_{\epsilon_j}}\}_{j \geq 0})$. For $m \in \mathbb{N}$ fixed, we let $\alpha(m)$ be such that p_{s+1}^m satisfies $\mathbf{P}_{r_m, \epsilon_{\alpha(m)}}$. Since $r_m \rightarrow 0$ as $m \rightarrow +\infty$, it is clear that $\alpha(m) \rightarrow +\infty$ as $m \rightarrow +\infty$. We set $\beta(m) = \min(m, \alpha(m))$. By \mathbf{H}_s , for $1 \leq i \leq s$, $p_i^{\alpha(m)}$ satisfies $\mathbf{P}_{r_{\alpha(m)}, \epsilon_{\alpha(m)}}$ and since r_m is decreasing, $p_i^{\alpha(m)}$ satisfies $\mathbf{P}_{r_{\beta(m)}, \epsilon_{\alpha(m)}}$. Moreover, p_{s+1}^m satisfies $\mathbf{P}_{r_m, \epsilon_{\alpha(m)}}$ and since r_m is decreasing p_{s+1}^m satisfies $\mathbf{P}_{r_{\beta(m)}, \epsilon_{\alpha(m)}}$. Up to the extraction of a subsequence, we can assume that $r_{\beta(m)} \searrow 0$ as $m \rightarrow +\infty$ and we let $p_{s+1} \in M$ such that $p_{s+1}^m \rightarrow p_{s+1}$ as $m \rightarrow +\infty$. Since $p_{s+1}^m \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$, $p_{s+1} \notin \{p_1, \dots, p_s\}$. By the same arguments as in the proof of \mathbf{H}_1 , we also have that $p_{s+1} \in \partial M$. This proves \mathbf{H}_{s+1} .

The proof of \mathbf{H}_{k+1} is complete. Now, we prove that \mathbf{H}_{k+1} leads to a contradiction. We define $k+1$ test functions for the variational characterization of $\sigma_{\epsilon_m} = \sigma_k(M, g, \partial M, e^{u_{\epsilon_m}})$, η_i^m for $m \in \mathbb{N}$ and $1 \leq i \leq k+1$ as follows

- If p_i^m satisfies $\mathbf{A}_{r_m, \epsilon_m}$, η_i^m is the extension by 0 in $M \setminus B_g(p_i^m, r_m)$ of an eigenfunction for $\sigma_*(B_g(p_i^m, r_m), g, I_g(p_i^m, r_m), \{e^{u_{\epsilon_m}}\})$. In this case,

$$\frac{\int_M |\nabla \eta_i^m|_g^2 dv_g}{\int_{\partial M} (\eta_i^m)^2 d\sigma_g} \leq \frac{\sigma_k(M, [g])}{2}. \quad (6.29)$$

- If p_i^m does not satisfies $\mathbf{A}_{r_m, \epsilon_m}$, it satisfies $\mathbf{B}_{r_m, \epsilon_m}$ and η_i^m is some eigenfunction for $\sigma_*(D_i^m, g, \Gamma_i^m, e^{u_{\epsilon_m}})$ extended by 0 in $M \setminus D_i^m$ where D_i^m is a nodal domain of some Steklov eigenfunction associated to σ_{ϵ_m} which is included in $B_g(p_i^m, r_m)$. Such a domain exists by assumption $\mathbf{B}_{r_m, \epsilon_m}$ and satisfies $\Gamma_i^m = \partial M \cap D_i^m \neq \emptyset$. In this case,

$$\frac{\int_M |\nabla \eta_i^m|_g^2 dv_g}{\int_{\partial M} (\eta_i^m)^2 d\sigma_g} = \sigma_*(D_i^m, g, \Gamma_i^m, e^{u_{\epsilon_m}}) = \sigma_{\epsilon_m}. \quad (6.30)$$

For m large enough, we have

$$\min_{1 \leq i < i' \leq k+1} d_g(p_i^m, p_{i'}^m) - 3r_m \geq \frac{1}{2} \min_{1 \leq i < i' \leq k+1} d_g(p_i, p_{i'}) > 0$$

so that the functions $\eta_1^m, \dots, \eta_{k+1}^m$ have pairwise disjoint supports. Thanks to (6.29) and (6.30), the min-max characterization of $\sigma_{\epsilon_m} = \sigma(M, g, \partial M, e^{u_{\epsilon_m}})$ (6.4) gives that

$$\sigma_{\epsilon_m} \leq \max_{1 \leq i \leq k+1} \frac{\int_M |\nabla \eta_i^m|_g^2 dv_g}{\int_{\partial M} (\eta_i^m)^2 d\sigma_g} \leq \sigma_{\epsilon_m}$$

since for m large enough, $\sigma_{\epsilon_m} \rightarrow \sigma_k(M, [g]) > \frac{\sigma_k(M, [g])}{2}$. Then, all the inequalities are equalities and by the case of equality in the min-max characterization of the k -th eigenvalue, one of the functions η_i^m is an eigenfunction on the surface for $\sigma_{\epsilon_m} = \sigma_k(M, g, \partial M, e^{u_{\epsilon_m}})$. Since $\text{supp}(\eta_i^m) \subset B_g(p_i^m, r_m)$ and $\eta_i^m \neq 0$, we contradict the harmonicity of η_i^m .

Therefore, we have proved that there exists a subsequence $\{e^{u_{\epsilon_m}}\}$ and $p_1, \dots, p_s \in \partial M$ for some $0 \leq s \leq k$ such that

$$\forall \rho > 0, \exists r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \subset \bigcap_{i=1}^s B_g(p_i, \rho),$$

which is exactly the first part of the claim.

Let's prove now the second part of the claim. If there exists a subsequence $m(j) \rightarrow +\infty$ as $j \rightarrow +\infty$ such that there exists $\rho > 0$ and $r > 0$ and $1 \leq i_0 \leq s$ with

$$X_r(M, g, \partial M, \{e^{u_{\epsilon_{m(j)}}}\}) \cap B_g(p_{i_0}, \rho) = \emptyset$$

then, taking the subsequence $m(j)$, we can remove the index $i_0 \in \{1, \dots, s\}$ so that

$$X_r(M, g, \partial M, \{e^{u_{\epsilon_{m(j)}}}\}) \subset \bigcup_{i \in \{1, \dots, s\} \setminus \{i_0\}} B_g(p_i, \rho).$$

We go on with this process until we cannot find a subsequence such that (6.27) does not hold. This ends the proof of the claim.

◇

Up to the extraction of a subsequence, we assume in the following that $\{e^{u_\epsilon}\}$ satisfies the conclusion of Claim 42. For $\rho > 0$, we let

$$M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$$

and

$$I(\rho) = \partial M \setminus \bigcup_{i=1}^s I_g(p_i, \rho) .$$

We are now able to get regularity estimates on the functions e^{u_ϵ} in $I(\rho)$ and Φ_ϵ in $M(\rho)$.

Claim 43. *We assume that $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{I(\rho)} e^{u_\epsilon} dv_g > 0$ for any $\rho > 0$ small enough. Then we have the following*

— Estimates on Φ_ϵ

$$\forall \rho > 0, \exists C_1(\rho) > 0, \forall \epsilon > 0, \|\Phi_\epsilon\|_{W^{1,2}(M(\rho))} \leq C_1(\rho) , \quad (6.31)$$

$$\forall \rho > 0, \exists C_2(\rho) > 0, \forall \epsilon > 0, \|\Phi_\epsilon\|_{C^0(M(\rho))} \leq C_2(\rho) , \quad (6.32)$$

— Quantitative non-concentration estimates on e^{u_ϵ} and $|\nabla \Phi_\epsilon|_g^2$

$$\forall \rho > 0, \exists D_1(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in I(\rho)} \int_{I_g(x, r)} e^{u_\epsilon} dv_g \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})} , \quad (6.33)$$

$$\forall \rho > 0, \exists D_2(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in I(\rho)} \int_{B_g(x, r)} |\nabla \Phi_\epsilon|_g^2 dv_g \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}} . \quad (6.34)$$

Proof. We first prove (6.31) by using Claim 42 and the assumption $m_0(\rho) > 0$.

For that purpose, let's prove that $\{\frac{e^{u_\epsilon}}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g}\}$ is bounded in $W^{-1,2}(M(\rho))$. Let $\rho > 0$ and let $r > 0$ be such that $X_r(M, g, \partial M, \{e^{u_\epsilon}\}) \subset \bigcup_{i=1}^s B_g(p_i, \rho)$. Then, for all $x \in I(\rho)$ and all $\epsilon > 0$, $\sigma_*(B_g(x, r), g, I_g(x, r), e^{u_\epsilon}) > \frac{\sigma_k(M, [g])}{2}$. By the compactness of $I(\rho)$, we can find $y_1, \dots, y_t \in I(\rho)$ such that

$$I(\rho) \subset \bigcup_{i=1}^t I_g(y_i, r) .$$

Let ψ_1, \dots, ψ_t be a partition of unity associated to this covering, such that $\sum_{i=1}^t \psi_i = 1$ on $I(\rho)$ and $\text{supp}(\psi_i) \subset B_g(y_i, r)$. Let $L : W^{1,2}(M(\rho)) \rightarrow W^{1,2}(M)$ be a continuous extension operator.

Then, if $\psi \in W^{1,2}(M(\rho))$, its trace on the boundary satisfies

$$\begin{aligned}
 \int_{I(\rho)} \psi \frac{e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g} &= \sum_{i=1}^t \int_{I(\rho) \cap B_g(y_i, r)} \psi \psi_i \frac{e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g} \\
 &\leq \sum_{i=1}^t \left(\int_{I(\rho) \cap B_g(y_i, r)} (\psi_i \psi)^2 \frac{e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^t \left(\int_{\partial M \cap B_g(y_i, r)} (\psi_i L(\psi))^2 \frac{e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g} \right)^{\frac{1}{2}} \\
 &\leq \sum_{i=1}^t \frac{\left(\int_{B_g(y_i, r)} |\nabla(\psi_i L(\psi))|_g^2 dv_g \right)^{\frac{1}{2}}}{\sigma_\star(B_g(y_i, r), g, I_g(y_i, r), e^{u_\epsilon} g)^{\frac{1}{2}} \left(\int_{I(\rho)} e^{u_\epsilon} d\sigma_g \right)^{\frac{1}{2}}} \\
 &\leq \frac{A_0(\rho)}{\left(\frac{\sigma_k(M, [g])}{2} \right)^{\frac{1}{2}} m_0(\rho)^{\frac{1}{2}}} \|L(\psi)\|_{W^{1,2}(M)} \\
 &\leq A_1(\rho) \|\psi\|_{W^{1,2}(M(\rho))}
 \end{aligned}$$

for some constants $A_0(\rho)$ and $A_1(\rho)$ which do not depend on $\epsilon > 0$.

By Theorem 16 in Section 6.2.3, we now get the following Poincaré inequality : there exists some constant $A_2(\rho)$ such that for any $f \in C^\infty(M(\rho))$

$$\forall \epsilon > 0, \int_{M(\rho)} \left(f - \int_{I(\rho)} f \frac{e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g} \right)^2 dv_g \leq A_2(\rho) \int_{M(\rho)} |\nabla f|_g^2 dv_g.$$

We deduce from this inequality that

$$\int_{M(\rho)} f^2 dv_g \leq 2A_2(\rho) \int_{M(\rho)} |\nabla f|_g^2 dv_g + 2V_g(M) \frac{\int_{I(\rho)} f^2 e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g}$$

Applying this inequality to the ϕ_ϵ^i 's and summing for $i = 0 \cdots n$, we get that

$$\int_{M(\rho)} |\Phi_\epsilon|^2 dv_g \leq 2A_2(\rho) \sigma_\epsilon \int_{\partial M} |\Phi_\epsilon|^2 d\sigma_g + 2V_g(M) \frac{\int_{\partial M} |\Phi_\epsilon|^2 e^{u_\epsilon} d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g}$$

using the fact that

$$\int_{M(\rho)} |\nabla \phi_\epsilon^i|_g^2 dv_g \leq \int_M |\nabla \phi_\epsilon^i|_g^2 dv_g = \sigma_\epsilon \int_{\partial M} e^{u_\epsilon} (\phi_\epsilon^i)^2 d\sigma_g.$$

by (iii) of proposition 6,

$$\int_{\partial M} e^{u_\epsilon} |\Phi_\epsilon|^2 d\sigma_g = \int_{\partial M} |\Phi_\epsilon|^2 K_\epsilon[v_\epsilon] d\sigma_g = \int_{\partial M} K_\epsilon[|\Phi_\epsilon|^2] dv_\epsilon = 1.$$

Then, we get that

$$\int_{M(\rho)} |\Phi_\epsilon|^2 dv_g \leq 2A_2(\rho) \sigma_\epsilon + \frac{2V_g(M)}{\int_{I(\rho)} e^{u_\epsilon} d\sigma_g}.$$

Thanks to the assumption of the claim, namely that $\int_{I(\rho)} e^{u_\epsilon} d\sigma_g \rightarrow m_0(\rho) > 0$, we get the existence of some $A_3(\rho)$ such that

$$\int_{M(\rho)} |\Phi_\epsilon|^2 dv_g \leq A_3(\rho) .$$

Now, with what we just said, we also know that

$$\int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g \leq \sigma_\epsilon$$

and (6.31) follows.

In order to get (6.32), we first prove that

$$\forall \rho > 0, \exists C_0(\rho), \forall \epsilon > 0, \|\Phi_\epsilon\|_{C^0(I(\rho))} \leq C_0(\rho) . \quad (6.35)$$

Let $\rho > 0, 0 \leq i \leq n$ and up to change ϕ_ϵ^i into $-\phi_\epsilon^i$, let (x_ϵ) be a sequence of points of $I(\rho)$ such that $\phi_\epsilon^i(x_\epsilon) = \sup_{I(\rho)} |\phi_\epsilon^i|$. We set

$$\delta_\epsilon = d_g(x_\epsilon, \text{supp}(v_\epsilon)) .$$

We divide the rest of the proof of (6.35) into three cases.

CASE 1 - We assume that $\delta_\epsilon^{-1} = O(1)$. Then, by (6.7), $\{e^{u_\epsilon}\}$ is uniformly bounded in $I_g(x_\epsilon, \min\{\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\})$. By (6.31), $\{\phi_\epsilon^i\}$ is bounded in $L^2(I(\frac{\rho}{2}))$. Then, in $W^{1,2}(I_g(x_\epsilon, \min\{\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\}))$, $\{\phi_\epsilon^i\}$ is bounded by elliptic theory for the Dirichlet-to-Neumann operator (see [109], chapter 7.11, page 37), and $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by Sobolev embeddings.

CASE 2 - We assume that $\delta_\epsilon = O(\sqrt{\epsilon})$. Then, $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by Claim 41.

CASE 3 - We assume that $\delta_\epsilon \rightarrow 0$ and $\frac{\sqrt{\epsilon}}{\delta_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. We let

$$\psi_\epsilon = \tilde{\phi}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon) \text{ for } x \in \mathbb{D}_{\delta\delta_\epsilon^{-1}}^+ \text{ and } e^{w_\epsilon} = \delta_\epsilon e^{\tilde{u}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon)} \text{ for } x \in I_{\delta\delta_\epsilon^{-1}}$$

so that

$$\begin{cases} \Delta \psi_\epsilon = 0 & \text{in } \mathbb{D}_{\delta\delta_\epsilon^{-1}}^+ \\ \partial_t \psi_\epsilon = -\sigma_\epsilon e^{w_\epsilon} \psi_\epsilon & \text{on } I_{\delta\delta_\epsilon^{-1}} . \end{cases} \quad (6.36)$$

Let $y_\epsilon \in \text{supp}(v_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) = \delta_\epsilon$ and set $z_\epsilon = \frac{\tilde{y}_\epsilon - \tilde{x}_\epsilon}{\delta_\epsilon}$ so that $z_\epsilon \rightarrow z_0$ as $\epsilon \rightarrow 0$ up to the extraction of a subsequence. We set $R = |z_0|$. Thanks to Claim 41, we know that $\psi_\epsilon(z_\epsilon) = \phi_\epsilon^i(y_\epsilon) = O(1)$. Thanks to estimates (6.9) on the heat kernel, there exists $D_1 > 0$ such that

$$e^{w_\epsilon} \leq D_1 \text{ on } I_{\frac{R}{2}} .$$

We first assume that ψ_ϵ does not vanish in \mathbb{D}_{3R}^+ . Then, we can apply Harnack's inequality and get some constant $D_2 > 0$ such that

$$\psi_\epsilon \geq D_2 \psi_\epsilon(0) \text{ on } \mathbb{D}_{\frac{R}{4}}^+$$

for all $\epsilon > 0$. Since ψ_ϵ is positive on $\mathbb{D}_{|z_\epsilon|}^+(z_\epsilon) \subset \mathbb{D}_{3R}^+$, by the equation (6.36), it is weakly superharmonic and we can write that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{\pi |z_\epsilon|} \int_{\partial \mathbb{D}_{|z_\epsilon|}^+(z_\epsilon)} \psi_\epsilon d\sigma.$$

Taking only the part of the integral which lies in $\mathbb{D}_{\frac{R}{4}}^+$, we get the existence of some constant $D_3 > 0$ such that

$$\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0)$$

and this concludes the proof of (6.32) in this case since $\phi_\epsilon^i(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now assume that ψ_ϵ vanishes on \mathbb{D}_{3R}^+ . Since $\delta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, and $x_\epsilon \in I(\rho)$, by Claim 42, ψ_ϵ vanishes on a piecewise smooth curve in \mathbb{D}_{4R}^+ which connects two points of distance greater than R . By the corollary of Theorem 17 of Section 6.2.3 on $\Omega = \mathbb{D}_{5R}^+$, we get some constant $C_R > 0$ such that

$$\int_{\mathbb{D}_{4R}^+} \psi_\epsilon^2 \leq C_R \int_{\mathbb{D}_{5R}^+} |\nabla \psi_\epsilon|^2 dx$$

which proves that $\{\psi_\epsilon\}$ is bounded in $W^{1,2}(\mathbb{D}_{4R}^+)$ by conformal invariance of the L^2 -norm of the gradient in dimension 2. By trace Sobolev properties, $\{\psi_\epsilon\}$ is bounded in $L^2(I_{4R})$ and by elliptic regularity theory for the Dirichlet-to-Neumann operator (see [109], Chapter 7.11, page 37), ψ_ϵ is bounded in $L^\infty(\mathbb{D}_{\frac{R}{4}})$ which gives that $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded.

The study of these three cases completes the proof of (6.35).

We now prove (6.32). Let $\rho > 0$ and $0 \leq i \leq n$. Then, since ϕ_ϵ^i is harmonic in $M(\frac{\rho}{2})$, by elliptic regularity theory, there exists a constant $K_0(\rho) > 0$ such that

$$\|\phi_\epsilon^i\|_{C^0(M(\rho))} \leq K_0(\rho) \left(\|\phi_\epsilon^i\|_{L^2(M(\frac{\rho}{2}))} + \|\phi_\epsilon^i\|_{C^0(I(\frac{\rho}{2}))} \right)$$

so that (6.32) holds with $C_2(\rho) = K_0(\rho) (C_1(\frac{\rho}{2}) + C_0(\frac{\rho}{2}))$.

Thanks to Claim 42, we have the existence of some $r_1(\rho) > 0$ such that for any $0 < r < r_1(\rho)$,

$$\forall \epsilon > 0, \forall x \in I(\rho), \frac{1}{\sigma_\star(B_g(x, r), g, I_g(x, r), e^{u_\epsilon} g)} \leq \frac{2}{\sigma_k(M, [g])}.$$

By isocapacity estimates,

$$\begin{aligned} \int_{I_g(x, r)} e^{u_\epsilon} d\sigma_g &\leq \frac{\text{Cap}_2(B_g(x, r), B_g(x, r_1))}{\sigma_\star(B_g(x, r), g, I_g(x, r), e^{u_\epsilon} g)} \\ &\leq 2 \frac{\text{Cap}_2(\mathbb{D}_{\frac{r}{C_0}}, \mathbb{D}_{C_0 r_1})}{\sigma_k(M, [g])} \\ &\leq \frac{4\pi}{\sigma_k(M, [g]) \ln\left(\frac{C_0^2 r_1}{r}\right)} \end{aligned}$$

and we get (6.33).

Finally, let's prove (6.34). We set for $x \in I_\delta$ such that $\bar{x} \in M(\rho)$, where $\bar{x} = \exp_{g_l, x_l}(x)$ as defined in the section 6.2.1 and for $0 < r \leq \delta$

$$F_\epsilon(r) = \int_{\mathbb{D}_r^+(x)} |\nabla \tilde{\Phi}_\epsilon|^2 dx .$$

We suppose in the following that $\delta < 1$, without loss of generality. We just aim at proving that

$$F_\epsilon(r) \leq \frac{D_0(\rho)}{\sqrt{\ln\left(\frac{1}{r}\right)}} .$$

We know that $\tilde{\Phi}_\epsilon$ satisfies the equations

$$\begin{cases} \Delta \tilde{\Phi}_\epsilon = 0 & \text{in } \mathbb{D}_\delta^+ \\ \partial_t \tilde{\Phi}_\epsilon = -\sigma_\epsilon e^{\tilde{u}_\epsilon} \tilde{\Phi}_\epsilon & \text{on } I_\delta \end{cases}$$

and we deduce that

$$F_\epsilon(r) = \sigma_\epsilon \int_{I_r(x)} e^{\tilde{u}_\epsilon} |\tilde{\Phi}_\epsilon|^2 dx + \int_{\partial \mathbb{D}_r^+(x)} \tilde{\Phi}_\epsilon \cdot \partial_\nu \tilde{\Phi}_\epsilon d\sigma_{\tilde{\zeta}} .$$

Using (6.32) and (6.33), there exist some constants $K_1(\rho)$ and $K_2(\rho)$ independent of ϵ, r and x with $\bar{x} \in I(\rho)$, such that

$$\begin{aligned} F_\epsilon(r)^2 &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + K_2(\rho) \left(\int_{\partial \mathbb{D}_r^+(x)} |\nabla \tilde{\Phi}_\epsilon| dx \right)^2 \\ &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + \pi r K_2(\rho) \int_{\partial \mathbb{D}_r^+(x)} |\nabla \tilde{\Phi}_\epsilon|^2 dx \\ &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + \pi r K_2(\rho) F'_\epsilon(r) . \end{aligned}$$

for any $0 < r < \delta$. We can write that

$$\begin{aligned} \left(F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} \right)'(s) &= F'_\epsilon(s) \sqrt{\ln\left(\frac{1}{s}\right)} - \frac{1}{2s \sqrt{\ln\left(\frac{1}{s}\right)}} F_\epsilon(s) \\ &\geq \frac{F_\epsilon(s)^2 \sqrt{\ln\left(\frac{1}{s}\right)}}{\pi s K_2(\rho)} - \frac{K_1(\rho)}{\pi s K_2(\rho) \ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} - \frac{1}{2s \sqrt{\ln\left(\frac{1}{s}\right)}} F_\epsilon(s) \end{aligned}$$

Setting

$$J_\epsilon = \left\{ s \in (0, \delta); F_\epsilon(s) < \frac{\pi K_2(\rho)}{\ln\left(\frac{1}{s}\right)} \right\} ,$$

we have for $s \in (0, \delta) \setminus J_\epsilon$

$$\left(F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} \right)'(s) \geq -\frac{K_3(\rho)}{s \ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} \quad (6.37)$$

for $K_3(\rho) = \frac{K_1(\rho)}{\pi K_2(\rho)}$. Let $r \in (0, \delta)$,

$$s_\epsilon = \inf\{s \in [r, \delta), s \in J_\epsilon\}$$

If $s_\epsilon = r$, then

$$F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} \leq \frac{\pi K_2(\rho)}{\sqrt{\ln\left(\frac{1}{r}\right)}} \leq \frac{\pi K_2(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}}$$

and if $s_\epsilon > r$, then, integrating (6.37) from r to s_ϵ leads to

$$\begin{aligned} F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} &\leq F_\epsilon(s_\epsilon) \sqrt{\ln\left(\frac{1}{s_\epsilon}\right)} + \int_r^{s_\epsilon} \frac{K_3(\rho)}{s \ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} ds \\ &\leq F_\epsilon(s_\epsilon) \sqrt{\ln\left(\frac{1}{s_\epsilon}\right)} + \frac{2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{s_\epsilon}\right)}}. \end{aligned}$$

If $s_\epsilon < \delta$, we deduce from this inequality and the definition of s_ϵ that

$$F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} \leq \frac{\pi K_2(\rho) + 2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}}$$

and if $s_\epsilon = \delta$,

$$F_\epsilon(r) \sqrt{\ln\left(\frac{1}{r}\right)} \leq \sigma_\epsilon \sqrt{\ln\left(\frac{1}{\delta}\right)} + \frac{2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}}$$

where we used conformal invariance of the L^2 -norm of the gradient to get $F_\epsilon(\delta) \leq \sigma_\epsilon$.

Gathering all the cases, we get (6.34) and this ends the proof of the claim. \diamond

In the following claim, we aim at passing to the limit in the equation (i) and the condition (ii) given by Proposition 6. The limiting functions would then satisfy (6.41) and (6.42).

Claim 44. *We assume that $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{I(\rho)} e^{2u_\epsilon} dv_g > 0$ for any $\rho > 0$ small enough. Then, the following assertions hold*

— *For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that*

$$\forall x \in I(\rho), |\Phi_\epsilon|^2(x) \geq 1 - \beta_\epsilon. \quad (6.38)$$

— *For $\rho > 0$ and $x \in I(\rho)$, we set $\Psi_\epsilon(x) = \frac{\Phi_\epsilon(x)}{|\Phi_\epsilon(x)|}$. Then for any $\rho > 0$, $\{\Psi_\epsilon\}$ is uniformly equicontinuous on $\mathcal{C}^0(I(\rho), \mathbb{S}^n)$.*

— *For any $\rho > 0$, up to the extraction of a subsequence of $\{\Phi_\epsilon\}$, there exist functions $\Phi \in W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \cap L^\infty(I(\rho), \mathbb{R}^{n+1})$ and $\Psi \in W^{\frac{1}{2},2}(I(\rho), \mathbb{S}^n) \cap \mathcal{C}^0(I(\rho), \mathbb{S}^n)$ such that*

$$\Phi_\epsilon \rightharpoonup \Phi \text{ in } W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \text{ as } \epsilon \rightarrow 0 \quad (6.39)$$

and

$$\Psi_\epsilon \rightarrow \Psi \text{ in } \mathcal{C}^0(I(\rho), \mathbb{S}^n) \text{ as } \epsilon \rightarrow 0 \quad (6.40)$$

with

$$|\Phi|^2 \geq_{a.e.} 1 \text{ and } \Psi = \frac{\Phi}{|\Phi|} \text{ on } I(\rho). \quad (6.41)$$

Moreover, for $0 \leq i \leq n$,

$$\begin{cases} \Delta_g \phi^i = 0 & \text{in } M(\rho) \\ \partial_\nu \phi^i = \sigma_k(M, [g]) \psi^i d\nu & \text{on } I(\rho) \end{cases} \quad (6.42)$$

in a weak sense.

Proof.

STEP 1 - Let $1 \leq i \leq s$. We prove that at the neighbourhood of the singular points defined in Claim 42,

$$\sup_{x \in I(\rho)} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) d\sigma_g(y) = O(e^{-\frac{\rho^2}{8\epsilon}}).$$

Let $x \in I(\rho)$. Then, by estimate (6.6) of Section 6.2.2

$$\begin{aligned} e^{\frac{\rho^2}{8\epsilon}} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) d\sigma_g(y) &\leq \frac{A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{31\rho^2}{400\epsilon}} \frac{\int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2 e^{u_\epsilon} d\sigma_g}{\inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon}} \\ &\leq \frac{A_0 e^{-\frac{31\rho^2}{400\epsilon}}}{\sqrt{4\pi\epsilon} \inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon}}, \end{aligned}$$

since by (iii) of Proposition 6,

$$\int_{\partial M} |\Phi_\epsilon|^2 e^{u_\epsilon} d\sigma_g = 1$$

We assume by contradiction that

$$\inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon} \leq \frac{e^{-\frac{31\rho^2}{400\epsilon}}}{\sqrt{\epsilon}}.$$

Let $y \in \overline{I_g(p_i, \frac{\rho}{10})}$ be such that $e^{u_\epsilon(y)} = \inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon}$. Then, by (6.6) of Section 6.2.2,

$$e^{u_\epsilon(y)} = \int_{\partial M} p_\epsilon(y, x) d\nu_\epsilon(x) \geq \frac{e^{-\left(\frac{2\rho}{10}\right)^2 \frac{1}{4\epsilon}}}{A_0 \sqrt{4\pi\epsilon}} \int_{I_g(p_i, \frac{\rho}{10})} d\nu_\epsilon$$

We deduce from this and the previous inequality that

$$\int_{I_g(p_i, \frac{\rho}{10})} d\nu_\epsilon \leq A_0 \sqrt{4\pi\epsilon} e^{-\frac{27\rho^2}{400\epsilon}}.$$

Let $z \in I_g(p_i, \frac{\rho}{20})$, and let us write thanks again to (6.6) of Section 6.2.2 that

$$e^{u_\epsilon(z)} \leq A_0 \frac{\int_{I_g(p_i, \frac{\rho}{10})} d\nu_\epsilon + e^{-\frac{\rho^2}{4\epsilon} \frac{1}{20^2}}}{\sqrt{4\pi\epsilon}} \leq \frac{A_0^2}{\sqrt{\epsilon}} e^{-\frac{27\rho^2}{400\epsilon}} + \frac{A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{\rho^2}{1600\epsilon}}.$$

Then, $\|e^{u_\epsilon}\|_{C^0(I_g(p_i, \frac{\rho}{20}))} \rightarrow 0$ as $\epsilon \rightarrow 0$. This implies that

$$\sigma_\star \left(B_g(p_i, \frac{\rho}{20}), g, I_g(p_i, \frac{\rho}{20}), e^{u_\epsilon} \right) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0 \quad (6.43)$$

It is clear that $\mathbf{A}_{\frac{\rho}{20}, \epsilon}$ defined before Claim 42 cannot be true for p_i and ϵ small enough. By (6.27) in Claim 42, $\mathbf{B}_{\frac{\rho}{20}, \epsilon}$ holds true for p_i . Then, there is an eigenfunction f associated to $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$ such that $f_\epsilon(p_i) = 0$ and the nodal set which contains p_i does not intersect $\partial B_g(p_i, \frac{\rho}{20}) \setminus \partial M$. We obtain a nodal domain $D_\epsilon \subset B_g(p_i, \frac{\rho}{10})$ for f_ϵ such that $p_i \in D_\epsilon \cap \partial M$. By 6.43,

$$\sigma_\epsilon = \sigma_\star(D_\epsilon, g, D_\epsilon \cap \partial M, e^{u_\epsilon}) \geq \sigma_\star \left(B_g(p_i, \frac{\rho}{20}), g, I_g(p_i, \frac{\rho}{20}), e^{u_\epsilon} \right) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Since $\sigma_\epsilon \leq \sigma_k(M, [g])$, we get a contradiction. This completes the proof of Step 1.

STEP 2 - There exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x, y \in I(\rho), \quad d_g(x, y) \leq \frac{\sqrt{\epsilon}}{\beta_\epsilon} \Rightarrow |\Phi_\epsilon(x) - \Phi_\epsilon(y)| \leq \beta_\epsilon. \quad (6.44)$$

We set $\gamma_\epsilon = \|\sqrt{\epsilon}e^{u_\epsilon}\|_{L^\infty(I(\rho))}^{\frac{1}{2}}$. We have $\gamma_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. Indeed, for $r > 0$, and $x \in I(\rho)$ such that $\gamma_\epsilon^2 = \sqrt{\epsilon}e^{u_\epsilon(x)}$,

$$\sqrt{\epsilon}e^{u_\epsilon(x)} \leq \frac{A_0}{\sqrt{4\pi}} \int_{I_g(x, r)} dv_\epsilon + o(1) = \frac{A_0}{\sqrt{4\pi}} \nu(I_g(x, r)) + o(1) \leq \frac{A_0 D_1(\rho)}{\sqrt{4\pi} \ln(\frac{1}{r})} + o(1)$$

By estimate (6.6), since $\nu_\epsilon \rightarrow_\star \nu$ as $\epsilon \rightarrow 0$ and by (6.33) of Claim 43. Letting $\epsilon \rightarrow 0$ and then $r \rightarrow 0$, we get $\gamma_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. We also have that $\frac{\gamma_\epsilon}{\sqrt{\epsilon}} \rightarrow +\infty$ as $\epsilon \rightarrow 0$, since $\gamma_\epsilon \geq \frac{m_0(\rho)}{2} \epsilon^{\frac{1}{4}}$ (indeed $m_0(\rho) + o(1) = \|e^{u_\epsilon}\|_{L^1(I(\rho))} \leq \|e^{u_\epsilon}\|_{L^\infty(I(\rho))}$). Let now $x_\epsilon, y_\epsilon \in I(\rho)$ with $d_g(x_\epsilon, y_\epsilon) \leq \frac{\sqrt{\epsilon}}{\gamma_\epsilon}$. Up to the extraction of a subsequence, $x_\epsilon \in \gamma_l$ for some l fixed and we set

$$\begin{cases} \hat{\Phi}_\epsilon(x) = \tilde{\Phi}^l(\tilde{x}_\epsilon^l + \frac{\sqrt{\epsilon}}{\gamma_\epsilon}x) \\ e^{\hat{u}_\epsilon(x)} = \frac{\sqrt{\epsilon}}{\gamma_\epsilon} e^{\tilde{u}_\epsilon(\tilde{x}_\epsilon^l + \frac{\sqrt{\epsilon}}{\gamma_\epsilon}x)} \end{cases}$$

which satisfy

$$\begin{cases} \Delta_{\tilde{\zeta}} \hat{\Phi}_\epsilon = 0 & \text{in } \mathbb{D}_{3C_0}^+ \\ \partial_t \hat{\Phi}_\epsilon = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\Phi}_\epsilon & \text{on } I_{3C_0} \end{cases} \quad (6.45)$$

Let α_ϵ be the mean value of $\hat{\Phi}_\epsilon$ in $\mathbb{D}_{3C_0}^+$. Then

$$\begin{aligned} \|\hat{\Phi}_\epsilon - \alpha_\epsilon\|_{L^\infty(I_{2C_0}(0))} &\leq D_0 \|\hat{\Phi}_\epsilon - \alpha_\epsilon\|_{H^1(I_{2C_0})} \\ &\leq D \|\partial_t \hat{\Phi}_\epsilon\|_{L^2(I_{3C_0})(\rho)} + D \|\hat{\Phi}_\epsilon - \alpha_\epsilon\|_{L^2(\mathbb{D}_{3C_0}^+(0))} \\ &\leq D\sigma_\epsilon \|\Phi_\epsilon\|_{L^\infty} C_0 \gamma_\epsilon + D' \|\nabla \hat{\Phi}_\epsilon\|_{L^2(\mathbb{D}_{3C_0}^+(0))} \\ &\leq D\sigma_\epsilon C_2(\rho) C_0 \gamma_\epsilon + \frac{D' \sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3C_0^2 \sqrt{\epsilon}}\right)^{\frac{1}{4}}}. \end{aligned}$$

The first inequality comes from Sobolev embeddings, the second comes from the regularity theory for the Dirichlet-to-Neumann operator (see [109], Chapter 7.11, page 37) looking at (6.45). The third inequality comes from the classical Poincaré inequality on $\mathbb{D}_{3C_0}^+$, and finally we use (6.32) and (6.34) in Claim 43. Setting

$$\beta_\epsilon = 2D\sigma_k(M, [g])C_2(\rho)C_0\gamma_\epsilon + \frac{2D'\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3C_0^2\sqrt{\epsilon}}\right)^{\frac{1}{4}}},$$

we have that $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and that

$$|\Phi_\epsilon(x_\epsilon) - \Phi_\epsilon(y_\epsilon)| \leq \beta_\epsilon.$$

Up to increase β_ϵ so that $\frac{\sqrt{\epsilon}}{\beta_\epsilon} \leq \frac{\sqrt{\epsilon}}{\gamma_\epsilon}$ we proved Step 2.

STEP 3 - For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x \in I(\rho), \left| |\Phi_\epsilon|^2(x) - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon \quad (6.46)$$

and

$$\forall x \in I(\rho) \cap \text{supp}(\nu_\epsilon), |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon. \quad (6.47)$$

Note that (6.46) implies (6.38) by Proposition 6. Let's prove (6.46). For $x \in I(\rho)$,

$$\begin{aligned} \left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right|(x) &\leq \int_{I_g(x, \frac{\epsilon}{\beta_\epsilon})} \left| |\Phi_\epsilon|^2(x) - |\Phi_\epsilon|^2(y) \right| p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + 2C_2\left(\frac{\rho}{10}\right)^2 \int_{\partial M \setminus I_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x, y) d\sigma_g \\ &\quad + \sum_{i=1}^s \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2(y) p_\epsilon(x, y) d\sigma_g(y). \end{aligned}$$

Notice that we can assume here that $\frac{\sqrt{\epsilon}}{\beta_\epsilon} \rightarrow 0$ up to increase β_ϵ and that we used (6.32). We can estimate the first RHS term thanks to Step 2 and (6.32), the second RHS term thanks to estimates (6.6) and the third RHS term thanks to Step 1 and we get

$$\left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right|(x) \leq 2C_2\left(\frac{\rho}{2}\right)\beta_\epsilon + O\left(e^{-\frac{1}{4C_0^4\beta_\epsilon^2}}\right) + O\left(e^{-\frac{\rho^2}{8\epsilon}}\right).$$

Up to increase β_ϵ , we get (6.46) and then (6.38).

Thanks to Point (iii) in Proposition (6), we deduce that

$$\forall x \in \text{supp}(\nu_\epsilon) \cap I(\rho), ||\Phi_\epsilon(x)| - 1| \leq \beta_\epsilon, \quad (6.48)$$

and for $x \in I(\rho)$, we have

$$\begin{aligned} ||\Phi_\epsilon| - K_\epsilon[|\Phi_\epsilon|]|(x) &\leq \int_{I_g(x, \frac{\epsilon}{\beta_\epsilon})} ||\Phi_\epsilon|(x) - |\Phi_\epsilon|(y)| p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + 2C_2\left(\frac{\rho}{10}\right) \int_{\partial M \setminus I_g(x, \frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x, y) d\sigma_g \\ &\quad + \sum_{i=1}^s \left(\int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2(y) p_\epsilon(x, y) d\sigma_g(y) \right)^{\frac{1}{2}}. \end{aligned}$$

and the same arguments, together with (6.48) lead to (6.47), up to increase again β_ϵ .

STEP 4 - Let $\Psi_\epsilon = \frac{\Phi_\epsilon}{|\Phi_\epsilon|}$ on $I(\rho)$. Then, for $\rho > 0$, there exists $C_3(\rho)$ such that

$$|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2 \sqrt{\ln \left(\frac{2\delta(\partial M)}{d_g(x, y)} \right)} \leq C_3(\rho)$$

for all $x, y \in I(\rho)$, where $\delta(\partial M)$ is the diameter of ∂M . In particular, Ψ_ϵ is uniformly equicontinuous on $I(\rho)$.

We first prove that there exists $D_3(\rho) > 0$ such that

$$\sup_{x \in I(\rho)} \sup_{v \in \Psi_\epsilon^\perp \cap \mathbb{S}^n} \frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq \frac{D_3(\rho)}{\sqrt{\ln \left(\frac{1}{r} \right)}} \quad (6.49)$$

for all r small enough. Indeed, for $x \in I(\rho)$ and $v \in \Psi_\epsilon(x)^\perp \cap \mathbb{S}^n$, $\Phi_\epsilon \cdot v$ vanishes at x . By Claim 42, x does not satisfy $\mathbf{B}_{r, \epsilon}$. Thus, the nodal set which contains x intersects $\partial B_g(x, r)$. By the corollary of Theorem 17 on a disc and a dilatation on this disc, we get some constant $D_4(\rho)$ such that

$$\frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq D_4(\rho) \int_{B_g(x, 2r)} |\nabla (\Phi_\epsilon \cdot v)|_g^2 dv_g$$

for all r small enough. With (6.34) in Claim 43, we deduce that

$$\frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_\epsilon \cdot v)^2 dv_g \leq \frac{D_2(\rho) D_4(\rho)}{\sqrt{\ln \left(\frac{1}{2r} \right)}}$$

for all r small enough. Thus, (6.49) is proved.

Assume now by contradiction that the conclusion of Step 4 is false : there exist $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$, x_m and y_m some points in $I(\rho)$ such that

$$|\Psi_{\epsilon_m}(x_m) - \Psi_{\epsilon_m}(y_m)|^2 \sqrt{\ln \left(\frac{1}{r_m} \right)} \rightarrow +\infty \text{ as } m \rightarrow +\infty \quad (6.50)$$

where $r_m = d_g(x_m, y_m) \rightarrow 0$ as $m \rightarrow +\infty$. Since for a fixed m , Ψ_{ϵ_m} is not constant at the neighbourhood of y_m , one can assume that for any m , $\Psi_{\epsilon_m}(y_m) \neq -\Psi_{\epsilon_m}(x_m)$ without changing (6.50). Thanks to (6.38), up to the extraction of a subsequence, there exists a fixed vector $v \in \mathbb{S}^n$ of the canonical basis of \mathbb{R}^{n+1} such that

$$\frac{1}{L_g(I_g(x_m, r_m))} \int_{I_g(x_m, r_m)} (\Phi_{\epsilon_m} \cdot v)^2 d\sigma_g \geq \frac{1}{n+1} + o(1).$$

Since, by Sobolev trace inequalities, there exists $K > 0$ independent of m such that

$$\begin{aligned} \frac{1}{L_g(I_g(x_m, r_m))} \int_{I_g(x_m, r_m)} (\Phi_{\epsilon_m} \cdot v)^2 d\sigma_g &\leq \frac{K}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_{\epsilon_m} \cdot v)^2 dv_g \\ &\quad + K \int_{B_g(x_m, r_m)} |\nabla (\Phi_{\epsilon_m} \cdot v)|_g^2 dv_g, \end{aligned}$$

we get thanks to (6.34) of Claim 43 that

$$\begin{aligned} \frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_{\epsilon_m} \cdot v)^2 dv_g &\geq \frac{1}{(n+1)K} - \frac{D_2(\rho)}{\sqrt{\ln\left(\frac{1}{r_m}\right)}} + o(1) \\ &= \frac{1}{K(n+1)} + o(1). \end{aligned}$$

Thanks to the assumption (6.50), we now prove that there exist $X_m \in \Psi_{\epsilon_n}(x_m)^\perp$ and $Y_m \in \Psi_{\epsilon_m}(y_m)^\perp$ such that

$$v = X_m + Y_m \text{ and } |X_m|^2 + |Y_m|^2 = o\left(\sqrt{\ln \frac{1}{r_m}}\right). \quad (6.51)$$

We denote $a_m = \Psi_{\epsilon_m}(x_m) \in \mathbb{S}^{k-1}$, $b_m = \Psi_{\epsilon_m}(y_m) \in \mathbb{S}^{k-1}$ and Π_m the vector space generated by a_m and b_m . Notice that Π_m is a plane since $b_m \notin \{a_m, -a_m\}$ by assumption. Let $c_m \in \Pi_m \cap \mathbb{S}^{k-1}$ such that $\{a_m, c_m\}$ is an orthonormal basis of Π_m . We get $\theta_m \in \mathbb{R}$ such that

$$b_m = \cos \theta_m a_m + \sin \theta_m c_m$$

and $\sin \theta_m \neq 0$. We let $v = p_m + q_m$ with $p_m \in \Pi_m$ and $q_m \in \Pi_m^\perp$. Notice that $|p_m| \leq 1$ and $|q_m| \leq 1$. Let $\alpha_m \in \mathbb{R}$ be such that

$$p_m = |p_m| (\cos \alpha_m a_m + \sin \alpha_m c_m).$$

We then set

$$\begin{aligned} X_m &= t_m c_m + q_m \in a_m^\perp \\ Y_m &= s_m (-\sin \theta_m a_m + \cos \theta_m c_m) \in b_m^\perp \end{aligned}$$

with

$$\begin{aligned} s_m &= -|p_m| \frac{\cos \alpha_m}{\sin \theta_m} \\ t_m &= |p_m| \left(\sin \alpha_m + \frac{\cos \alpha_m \cos \theta_m}{\sin \theta_m} \right) \end{aligned}$$

so that $v = X_m + Y_m$. Then,

$$|X_m|^2 + |Y_m|^2 = |q_m|^2 + t_m^2 + s_m^2 \leq 1 + f_{\theta_m}(\alpha_m),$$

where for α and $\theta \in \mathbb{R}$,

$$f_\theta(\alpha) = \frac{\cos^2 \alpha}{\sin^2 \theta} + \left(\sin \alpha + \frac{\cos \alpha \cos \theta}{\sin \theta} \right)^2 = \frac{1 + \cos^2 \theta \cos 2\alpha + \cos \theta \sin \theta \sin 2\alpha}{\sin^2 \theta}.$$

We easily prove that $f_\theta(\alpha) \leq f_\theta(\frac{\theta}{2}) = \frac{1}{1 - \cos \theta}$. Then,

$$|X_m|^2 + |Y_m|^2 \leq O\left(\frac{1}{1 - \cos \theta_m}\right) = O\left(\frac{1}{|a_m - b_m|^2}\right) = o\left(\sqrt{\ln \frac{1}{r_m}}\right).$$

This ends the proof of (6.51).

We now write thanks to (6.49) that

$$\begin{aligned}
 \frac{1}{(n+1)K} + o(1) &\leq \frac{1}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_{\epsilon_m} \cdot v)^2 dv_g \\
 &\leq \frac{2}{\text{Vol}_g(B_g(x, r))} \int_{B_g(x, r)} (\Phi_{\epsilon_m} \cdot X_m)^2 dv_g \\
 &\quad + \frac{2\text{Vol}_g(B_g(y_m, 2r_m))}{\text{Vol}_g(B_g(x_m, r_m))} \frac{1}{\text{Vol}_g(B_g(y_m, 2r_m))} \int_{B_g(y_m, 2r_m)} (\Phi_{\epsilon_m} \cdot Y_m)^2 dv_g \\
 &\leq 2D_3(\rho) |X_m|^2 \left(\ln \left(\frac{1}{r_m} \right) \right)^{-\frac{1}{2}} + 8C_0^2 D_3(\rho) |Y_m|^2 \left(\ln \left(\frac{1}{2r_m} \right) \right)^{-\frac{1}{2}} \\
 &= o(1).
 \end{aligned}$$

This clearly gives a contradiction and proves Step 4.

It is clear now that there exists some functions Φ and Ψ such that up to the extraction of a subsequence, (6.39), (6.40) and (6.41) hold. It remains to prove Step 5 :

STEP 5 - We have that

$$\phi_\epsilon^i e^{u_\epsilon} d\sigma_g \rightharpoonup_\star \psi^i dv \text{ as } \epsilon \rightarrow 0 \text{ in } I(\rho).$$

Let $\zeta \in C_c^0(I(\rho))$. Then

$$\begin{aligned}
 \int_{\partial M} \zeta \phi_\epsilon^i e^{2u_\epsilon} d\sigma_g - \int_{\partial M} \zeta \psi^i dv &= \int_{\partial M} \left(K_\epsilon[\zeta \phi_\epsilon^i] - \zeta K_\epsilon[\phi_\epsilon^i] \right) dv_\epsilon \\
 &\quad + \int_{\partial M} \zeta \left(K_\epsilon[\phi_\epsilon^i] - \psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] \right) dv_\epsilon \\
 &\quad + \int_{\partial M} \zeta \left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] - \psi_\epsilon^i \right) dv_\epsilon \\
 &\quad + \int_{\partial M} \zeta \left(\psi_\epsilon^i dv_\epsilon - \psi^i dv \right).
 \end{aligned} \tag{6.52}$$

Let us estimate these four terms. We have for $x \in \partial M$ that

$$\begin{aligned}
 \left| K_\epsilon[\zeta \phi_\epsilon^i] - \zeta K_\epsilon[\phi_\epsilon^i] \right| (x) &= \left| \int_{\partial M} (\zeta(y) - \zeta(x)) \phi_\epsilon^i(y) p_\epsilon(x, y) d\sigma_g(y) \right| \\
 &\leq C_2 \left(\frac{\rho}{10} \right) \int_{I(\frac{\rho}{10})} |\zeta(y) - \zeta(x)| p_\epsilon(x, y) d\sigma_g(y) \\
 &\quad + |\zeta(x)| \sum_{j=1}^s \int_{I_g(p_j, \frac{\rho}{10})} \left| \phi_\epsilon^i(y) \right| p_\epsilon(x, y) d\sigma_g(y)
 \end{aligned}$$

since $\text{supp}(\zeta) \subset I(\rho)$ and thanks to (6.32) of Claim 43. By Step 1 and since $\text{supp}(\zeta) \subset I(\rho)$, we deduce that this function uniformly converges to 0 in ∂M as $\epsilon \rightarrow 0$. Thus, the first RHS term

in (6.52) converges to 0 as $\epsilon \rightarrow 0$. For $x \in I(\rho)$,

$$\begin{aligned} \left| K_\epsilon[\phi_\epsilon^i] - \psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] \right| (x) &\leq \int_{\partial M} \left| \phi_\epsilon^i(y) - \psi_\epsilon^i(x) |\phi_\epsilon|(y) \right| p_\epsilon(x, y) d\sigma_g(y) \\ &\leq \int_{I(\frac{\rho}{10})} |\Phi_\epsilon(y)| \left| \psi_\epsilon^i(y) - \psi_\epsilon^i(x) \right| p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + 2 \sum_{j=1}^s \int_{I_g(p_j, \frac{\rho}{10})} |\Phi_\epsilon(y)| p_\epsilon(x, y) d\sigma_g(y) \\ &\leq C_2 \left(\frac{\rho}{10} \right) \int_{I(\frac{\rho}{10})} \left| \psi_\epsilon^i(y) - \psi_\epsilon^i(x) \right| p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + O(e^{-\frac{\rho^2}{16\epsilon}}). \end{aligned}$$

thanks to (6.32) of Claim 43 and Step 1. Thanks to the uniform equicontinuity of $\{\Psi_\epsilon\}$ on $I(\frac{\rho}{10})$, it uniformly converges to zero in ∂M as $\epsilon \rightarrow 0$. Thus, the second RHS term of (6.52) converges to 0 as $\epsilon \rightarrow 0$. Thanks to (6.47), we can write since $|\Psi_\epsilon| = 1$ that

$$\left| \int_{\partial M} \zeta \left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|] - \psi_\epsilon^i \right) dv_\epsilon \right| \leq \beta_\epsilon \|\zeta\|_\infty$$

so that the third RHS term in (6.52) converges to 0 as $\epsilon \rightarrow 0$. At last, we use the convergences $\Psi_\epsilon \rightarrow \Psi$ in $\mathcal{C}^0(I(\rho))$ and $\nu_\epsilon \rightarrow_\star \nu$ on $I(\rho)$ to obtain that the fourth RHS term in (6.52) also converges to 0 as $\epsilon \rightarrow 0$. This clearly ends the proof of Step 5.

Finally, passing to the weak limit in $I(\rho)$ for $\rho > 0$, in the equation satisfied by ϕ_ϵ^i permits to end the proof of the claim thanks to these steps.

◇

Thanks to Claim 44, with the assumption $m_0(\rho) = \lim_{\epsilon \rightarrow 0} \int_{I(\rho)} e^{u_\epsilon} dv_g > 0$, a diagonal extraction gives some functions $\Phi : M \setminus \{p_1, \dots, p_s\} \rightarrow \mathbb{R}^{n+1}$ and $\Psi : \partial M \setminus \{p_1, \dots, p_s\} \rightarrow \mathbb{S}^n$ such that for all $\rho > 0$ the conclusions (6.39), (6.40), (6.41) and (6.42) hold true for Φ and Ψ .

6.4.2 Energy estimates

Now, we give some energy estimates which will be useful later in the proof. We set a function ω on M satisfying the following equation

$$\begin{cases} \Delta_g \omega = 0 & \text{in } M \\ \omega = |\Phi| & \text{on } \partial M \end{cases} \quad (6.53)$$

in a weak sense. Since $|\Phi| \in W^{\frac{1}{2},2}(\partial M)$, such a solution exists and satisfies $\omega \in W^{1,2}(M)$ (see [43], Theorem 8.3). Let's prove this energy inequality :

Claim 45.

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g \geq \int_M \frac{|\nabla \Phi|_g^2}{\omega} dv_g \geq \sigma_k(M, [g]) m + \int_M \frac{|\Phi|^2 |\nabla \omega|_g^2}{\omega^3} dv_g \quad (6.54)$$

where $m = \lim_{\rho \rightarrow 0} m_0(\rho) = \lim_{\rho \rightarrow 0} \int_{I(\rho)} dv$.

Proof. Let $\rho > 0$. By Claim 40, there exists $C > 0$ independent of ρ and a nonnegative function $\eta \in \mathcal{C}^\infty(M)$ such that $\text{supp}(\eta) \subset M(\rho)$, $\eta = 1$ on $M(\sqrt{\rho})$, $0 \leq \eta \leq 1$, and

$$\int_M |\nabla \eta|_g^2 dv_g \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

By the weak maximum principle on (6.53), (see [43], Theorem 8.1),

$$\inf_M \omega \geq \inf_{\partial M} |\Phi| \geq 1$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g &\geq \int_{M(\rho)} |\nabla \Phi|_g^2 dv_g \\ &\geq \int_M \eta \frac{|\nabla \Phi|_g^2}{\omega} dv_g \\ &= \sum_{i=0}^n \int_M \left\langle \nabla \frac{\eta \phi_i}{\omega}, \nabla \phi_i \right\rangle_g dv_g \\ &\quad - \sum_{i=0}^n \int_M \frac{\phi_i}{\omega} \langle \nabla \eta, \nabla \phi_i \rangle_g dv_g \\ &\quad - \sum_{i=0}^n \int_M \phi_i \eta \left\langle \nabla \frac{1}{\omega}, \nabla \phi_i \right\rangle_g dv_g. \end{aligned}$$

We have that

$$\begin{aligned} \sum_{i=0}^n \int_M \left\langle \nabla \frac{\eta \phi_i}{\omega}, \nabla \phi_i \right\rangle_g dv_g &= \sum_{i=0}^n \int_M \frac{\eta \phi_i}{\omega} \Delta_g \phi_i dv_g + \sum_{i=0}^n \int_{\partial M} \frac{\eta \phi_i}{\omega} \partial_\nu \phi_i d\sigma_g \\ &= \sigma_k(M, [g]) \int_{\partial M} \eta \frac{|\Phi|}{\omega} d\nu \\ &= \sigma_k(M, [g]) \int_{\partial M} \eta d\nu \end{aligned}$$

thanks to (6.42) and that

$$\begin{aligned} \sum_{i=0}^n \int_M \eta \phi_i \left\langle \nabla \frac{1}{\omega}, \nabla \phi_i \right\rangle_g dv_g &= - \int_M \left\langle \nabla \eta, \nabla \frac{1}{\omega} \right\rangle_g \frac{|\Phi|^2}{2} dv_g \\ &\quad + \int_M \eta \frac{|\Phi|^2}{2} \Delta_g \left(\frac{1}{\omega} \right) dv_g + \int_{\partial M} \frac{|\Phi|^2}{2} \eta \partial_\nu \left(\frac{1}{\omega} \right) d\sigma_g \\ &= \int_M \langle \nabla \eta, \nabla \omega \rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g \\ &\quad - \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g - \frac{1}{2} \int_M |\Phi|^2 \eta \frac{\Delta_g \omega}{\omega^2} dv_g \\ &\quad - \frac{1}{2} \int_{\partial M} \eta \partial_\nu \omega d\sigma_g \\ &= \int_M \langle \nabla \eta, \nabla \omega \rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g - \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g \\ &\quad + \frac{1}{2} \int_M \eta \Delta_g \omega dv_g - \frac{1}{2} \int_M \langle \nabla \eta, \nabla \omega \rangle_g dv_g \end{aligned}$$

so that

$$\begin{aligned}
 \lim_{\epsilon \rightarrow 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g &\geq \int_{M(\rho)} |\nabla \Phi|_g^2 dv_g \\
 &\geq \sigma_k(M, [g]) \int_{\partial M} \eta dv + \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g \\
 &\quad - \sum_{i=0}^n \int_M \frac{\phi_i}{\omega} \langle \nabla \eta, \nabla \phi_i \rangle_g dv_g \\
 &\quad - \int_M \langle \nabla \eta, \nabla \omega \rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g + \frac{1}{2} \int_M \langle \nabla \eta, \nabla \omega \rangle_g dv_g \\
 &\geq \sigma_k(M, [g]) \int_{\partial M} \eta dv + \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g - \frac{C'}{\sqrt{\ln\left(\frac{1}{\rho}\right)}}
 \end{aligned}$$

where C' is a constant independent of ρ . Indeed, $\phi_i, \omega \in W^{1,2}(M)$ and we have for $0 \leq i \leq n$ that

$$\Delta_g(\omega - \phi_i) = 0 \text{ and } \Delta_g(\omega + \phi_i) = 0$$

in a weak sense. By the weak maximum principle (see [43], Theorem 8.1),

$$\inf_M (\omega - \phi_i) \geq \inf_{\partial M} (\omega - \phi_i) \geq 0$$

and

$$\inf_M (\omega + \phi_i) \geq \inf_{\partial M} (\omega + \phi_i) \geq 0$$

since $|\phi_i| \leq |\Phi| \leq \omega$ on ∂M . Then,

$$\sup_M \frac{|\phi_i|}{\omega} \leq 1 \text{ and } \sup_M \frac{|\Phi|^2}{\omega^2} \leq n + 1.$$

We finally get the claim, passing to the limit as $\rho \rightarrow 0$.

◇

6.5 Scales of concentration for the maximizing sequence

6.5.1 Concentration, capacity and rescalings

In this section, we aim at describing all the concentration scales of the sequence $\{e^{u_\epsilon} d\sigma_g\}$. We denote by $Z(M, \{e^{u_\epsilon} d\sigma_g\})$ the concentration points of a sequence of measures $\{e^{u_\epsilon} d\sigma_g\}$ on the boundary ∂M of a surface (M, g) that is

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) = \{z \in M; \lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \int_{I_g(z, r)} e^{u_\epsilon} d\sigma_g > 0\}.$$

Taking the maximizing sequence $\{e^{u_\epsilon} d\sigma_g\}$ for $\sigma_k(M, [g])$ given by the previous subsection, which converges to ν in $\mathcal{M}_1(\partial M)$, we clearly have that

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) = \{z \in \partial M; \nu(\{z\}) > 0\}$$

and that

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) \subset \bigcap_{r>0} X_r(M, \{e^{u_\epsilon} \sigma_g\}) = \{p_1, \dots, p_s\}, \quad (6.55)$$

where p_1, \dots, p_s are defined in Claim 42. This is a consequence of Claim 40 in Section 6.2.3 : indeed, for $x \in Z(M, \{e^{u_\epsilon} d\sigma_g\})$ and for $r > 0$ small enough, let η_{g,x,r,r^2} be given by Claim 40. Then

$$\sigma_\star(B_g(x, r), g, I_g(x, r), e^{u_\epsilon}) \leq \frac{\int_M \left| \nabla \eta_{g,x,r,r^2} \right|_g^2 dv_g}{\int_{\partial M} \left(\eta_{g,x,r,r^2} \right)^2 e^{u_\epsilon} d\sigma_g} \leq \frac{C}{\ln\left(\frac{1}{r}\right) \int_{B_g(x, r^2)} e^{u_\epsilon} d\sigma_g}$$

so that

$$\lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \sigma_\star(B_g(x, r), g, I_g(x, r), e^{u_\epsilon}) = 0.$$

Then there is a subsequence $\{\epsilon_j\}$ for which x satisfies $\mathbf{A}_{r, \epsilon_j}$ for all r small enough. Thanks to Claim 42, this gives that $x \in \{p_1, \dots, p_s\}$.

We now define some functions which will rescale the problem at the neighbourhood of the concentration points. For $a \in \mathbb{R} \times \{0\}$ and $\alpha > 0$, we let

$$H_{a, \alpha}(y) = \alpha y + a \text{ for } y \in \mathbb{R}^2.$$

For $p = (1, 0) \in \mathbb{S}^1$, we define $\lambda : \mathbb{D} \setminus \{p\} \rightarrow \mathbb{R}_+^2$ the conformal diffeomorphism such that

$$F \circ \lambda \circ F^{-1}(z) = i \frac{z+1}{1-z}$$

with its inverse

$$F \circ \lambda^{-1} \circ F^{-1}(z) = \frac{z-i}{z+i}$$

where $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ is the canonical map $F(x, y) = x + iy$. In this section, we prove the following :

Proposition 7. *There exist some points $a_1^\epsilon, \dots, a_N^\epsilon \in \mathbb{R} \times \{0\}$ and some scales*

$$0 < \alpha_N^\epsilon < \alpha_{N-1}^\epsilon < \dots < \alpha_1^\epsilon$$

such that for $1 \leq i \leq N$,

$$\alpha_i^\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (6.56)$$

and letting

$$F_i = \left\{ j > i; \frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon} \text{ is bounded} \right\},$$

we have for $j \neq i$ that

$$j \in F_i \Rightarrow \frac{\alpha_j^\epsilon}{\alpha_i^\epsilon} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (6.57)$$

and that

$$j \notin F_i \Rightarrow \frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0. \quad (6.58)$$

There are some disjoint sets $I_0^\epsilon, I_1^\epsilon, \dots, I_N^\epsilon \subset \partial M$, some sets $\Gamma_1^\epsilon, \dots, \Gamma_N^\epsilon \subset \mathbb{R} \times \{0\}$ and $S_1^\epsilon, \dots, S_N^\epsilon \subset \mathbb{S}^1$ given by

$$\Gamma_i^\epsilon = H_{a_i^\epsilon, \alpha_i^\epsilon}^{-1} \left(\tilde{\Gamma}_i^{\epsilon l_i} \right) \text{ and } S_i^\epsilon = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \circ \lambda \right)^{-1} \left(\tilde{\Gamma}_i^{\epsilon l_i} \right)$$

some associated densities defined by

$$e^{\tilde{u}_i^\epsilon} ds = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \right)^* \left(e^{\tilde{u}_i^{\epsilon l_i}} ds \right) \text{ and } e^{\tilde{u}_i^\epsilon} d\theta = \left(H_{a_i^\epsilon, \alpha_i^\epsilon} \circ \lambda \right)^* \left(e^{\tilde{u}_i^{\epsilon l_i}} ds \right)$$

some masses $m_i > 0$ satisfying

$$L_{e^{u_\epsilon} d\sigma_g}(I_i^\epsilon) = L_{e^{\tilde{u}_i^\epsilon} ds}(\Gamma_i^\epsilon) = L_{e^{\tilde{u}_i^\epsilon} d\theta}(S_i^\epsilon) \rightarrow m_i \text{ as } \epsilon \rightarrow 0 \quad (6.59)$$

for $1 \leq i \leq N$ and some $l_i \in \{1, \dots, L\}$, and $m_0 \geq 0$ satisfying

$$L_{e^{u_\epsilon} d\sigma_g}(I_0^\epsilon) \rightarrow m_0 \text{ as } \epsilon \rightarrow 0 \quad (6.60)$$

such that

$$Z(\mathbb{S}^1, \{\mathbf{1}_{S_i^\epsilon} e^{\tilde{u}_i^\epsilon} d\theta\}) = \emptyset \quad (6.61)$$

for $1 \leq i \leq N$,

$$Z(M, \{\mathbf{1}_{I_0^\epsilon} e^{u_\epsilon} d\sigma_g\}) = \emptyset \quad (6.62)$$

and

$$\sum_{i=0}^N m_i = 1. \quad (6.63)$$

6.5.2 Proof of Proposition 7

Let us denote by z_1, \dots, z_{N_0} the atoms of ν with $N_0 \leq s \leq k$ (s is given by 6.55 or Claim 42) so that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup^* \nu_0 + \sum_{i=1}^{N_0} m_i \delta_{z_i}$$

where $\nu_0 \in \mathcal{M}(\partial M)$ has no atoms. Let $m_0 = \int_{\partial M} d\nu_0 \geq 0$. All the m_i 's are positive for $1 \leq i \leq N_0$, and

$$\sum_{i=0}^{N_0} m_i = 1.$$

Let $1 \leq i \leq N_0$. We choose $l_i \in \{1, \dots, L\}$ such that $z_i \in \gamma_{l_i}$. Up to the extraction of a subsequence, one can build a sequence $\{r_i^\epsilon\}$ such that $r_i^\epsilon > 0$ and $r_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ with

$$\int_{I_g(z_i, r_i^\epsilon)} e^{u_\epsilon} d\sigma_g \rightarrow m_i \text{ as } \epsilon \rightarrow 0.$$

We associate to sequences $a_i^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_i^\epsilon > 0$ that we shall choose later the sets

$$\Gamma_i^\epsilon = H_{a_i^\epsilon, \alpha_i^\epsilon}^{-1} \left(\widetilde{I_g(z_i, r_i^\epsilon)}^{l_i} \right) \subset \mathbb{R} \times \{0\},$$

$$S_i^\epsilon = \lambda^{-1}(\Gamma_i^\epsilon) \subset \mathbb{S}^1,$$

$$M_i^\epsilon = B_g(z_i^\epsilon, r_i^\epsilon) ,$$

$$I_i^\epsilon = I_g(z_i^\epsilon, r_i^\epsilon) ,$$

$$M_0^\epsilon = M \setminus \bigcup_{i=1}^{N_0} M_i^\epsilon ,$$

$$I_0^\epsilon = \partial M \setminus \bigcup_{i=1}^{N_0} I_i^\epsilon ,$$

and the densities

$$e^{\hat{u}_i^\epsilon} = \alpha_i^\epsilon e^{(\hat{u}_i^\epsilon + \hat{v}_i^\epsilon) \circ H_{a_i^\epsilon, \alpha_i^\epsilon}} : \Gamma_i^\epsilon \rightarrow \mathbb{R} ,$$

$$e^{\hat{u}_i^\epsilon} d\theta = \lambda^*(e^{\hat{u}_i^\epsilon} ds) : S_i^\epsilon \rightarrow \mathbb{R} .$$

For the notations, we refer to Section 6.2.1.

Note that

$$M = M_0^\epsilon \cup \bigcup_{i=1}^{N_0} M_i^\epsilon$$

with $L_{e^{\hat{u}_i^\epsilon} d\sigma_g}(I_i^\epsilon) \rightarrow m_i$ as $\epsilon \rightarrow 0$ for $0 \leq i \leq N_0$. We assign to the subset M_i^ϵ a test function $\eta_i^\epsilon \in C_c^\infty(M_i^\epsilon)$ given by Claim 40 in Section 6.2.3

$$\eta_i^\epsilon = \eta_{g, z_i, (r_i^\epsilon)^{\frac{1}{2}}, r_i^\epsilon} \text{ for } 1 \leq i \leq N_0 ,$$

$$\eta_0^\epsilon = 1 - \sum_{i=1}^{N_0} \eta_{g, z_i, (r_i^\epsilon)^{\frac{1}{4}}, (r_i^\epsilon)^{\frac{1}{2}}} .$$

Note that these test functions with pairwise disjoint supports and small Rayleigh quotient may also be used to prove that $N_0 \leq k$ if $m_0 = 0$ or $N_0 \leq k - 1$ if $m_0 > 0$.

For $1 \leq i \leq N_0$, let's now adjust the parameters a_i^ϵ and α_i^ϵ in order to detect other scales of concentration of the mass at the neighbourhood of z_i . By Hersch theorem (see [54], lemma 1.1 in the case of the circle S^1) we can choose $a_i^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_i^\epsilon > 0$ such that

$$\int_{S^1} x e^{\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} d\theta = 0 . \quad (6.64)$$

Note that $\bar{a}_i^\epsilon \rightarrow z_i$ and that $\alpha_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. This normalization of the center of mass gives a dichotomy in the description of the concentration points of $\{e^{\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} d\theta\}$: if $z \in Z(S^1, \{e^{\hat{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} d\theta\})$, then, some mass is also concentrated in the opposite hemisphere $\{x \in S^1; (x, z) \leq 0\}$ and we can increase the number of test functions with small Rayleigh quotient on the manifold among $\eta_1^\epsilon, \dots, \eta_{N_0}^\epsilon$. From this remark, we will build by induction a finite bubble tree which describes the concentrations at all the scales they appear.

A tree T is a set of finite sequences

$$\gamma = (i_1, \dots, i_{|\gamma|}) \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$$

where $|\gamma|$ is the length of γ which satisfies

- $(\emptyset) \in T$ is the root of the tree
- if $\gamma \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$ and $i \in \mathbb{N}$, then $(\gamma, i) \in T \Rightarrow \gamma \in T$ and (γ, i) is called a son of γ .
- If $(\gamma, 0) \in T$ then $\forall i \in \mathbb{N}, (\gamma, 0, i) \notin T$. $(\gamma, 0)$ is called a leaf of T . We denote by L_T the set of leaves of T .
- If $\gamma \in T$, then $\{i \in \mathbb{N}; (\gamma, i) \in T\} = \{0, \dots, N_\gamma\}$ with $N_\gamma \in \mathbb{N}$ and N_γ is the number of sons of γ .

Let T be a tree. We let $|T| = \sup\{|\gamma|; \gamma \in T\}$ be the depth of the tree. We let also $T_j = \{\gamma \in T; |\gamma| \leq j\}$ be the truncated tree of depth $j \in \mathbb{N}$. We say that $\tilde{\gamma} \in T$ is a descendant of $\gamma \in T$ if there exists $\gamma' \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j$ such that $\tilde{\gamma} = (\gamma, \gamma')$.

In the following, we define by induction a tree T with

- some sets $I_\gamma^\epsilon \subset \partial M$ for $\gamma \in T$ and $\Gamma_\gamma^\epsilon \subset \mathbb{R} \times \{0\}$, $S_\gamma^\epsilon \subset \mathbb{S}^1$ for $\gamma \in T \setminus L_T$,
- some parameters $l_\gamma \in \{1, \dots, L\}$, $r_\gamma^\epsilon > 0$, $a_\gamma^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_\gamma^\epsilon > 0$ for $\gamma \in T \setminus L_T$,
- some points $z_\gamma \in \mathbb{S}^1$ if $\gamma \in T \setminus L_T$ and $|\gamma| \geq 2$ and $z_\gamma \in \partial M$ if $\gamma \in T \setminus L_T$ and $|\gamma| = 1$,
- some measures $\nu_0 \in \mathcal{M}(M)$ of mass $m_0 = \int_M d\nu_0 \geq 0$, $\nu_\gamma \in \mathcal{M}(\mathbb{S}^1)$ of mass $m_\gamma = \int_{\mathbb{S}^1} d\nu_\gamma \geq 0$ if $\gamma \in L_T$ and $|\gamma| \geq 2$ and some masses $m_\gamma > 0$ for $\gamma \in T \setminus L_T$,
- some functions $\hat{u}_\gamma^\epsilon : \Gamma_\gamma^\epsilon \rightarrow \mathbb{R}$ and $\check{u}_\gamma^\epsilon : S_\gamma^\epsilon \rightarrow \mathbb{R}$,
- some test functions $\eta_\gamma^\epsilon : M \rightarrow \mathbb{R}$ with $\eta_\gamma^\epsilon \in C_c^\infty(M_\gamma^\epsilon)$ for $\gamma \in T$

depending on ϵ . We describe the process of construction, by induction of this tree now and will prove in Claim 46 that it is a finite tree.

If $\gamma \in T$ and $|\gamma| = 1$, these objects are defined at the beginning of Section 6.5.2.

Assume now that these objects are defined for all γ of length $|\gamma| \leq j$. Let $\gamma \in T \setminus L_T$ with $|\gamma| \leq j$. Then, up to the extraction of a subsequence,

$$\mathbf{1}_{S_\gamma^\epsilon} e^{\check{u}_\gamma^\epsilon} d\theta \rightharpoonup^* \nu_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}} \quad (6.65)$$

where for $1 \leq i \leq N_\gamma$, $m_{(\gamma,i)} > 0$, $m_{(\gamma,0)} = \int_{\mathbb{S}^1} d\nu_{(\gamma,0)}$ and $\nu_{(\gamma,0)}$ is without atom. As we will see in the proof of Claim 46 and by the same arguments as in the previous subsection, Claim 40 provides some test functions which prove that $N_\gamma \leq k$. Notice that

$$\sum_{i=0}^{N_\gamma} m_{(\gamma,i)} = m_\gamma.$$

Let $1 \leq i \leq N_\gamma$. We define $l_{(\gamma,i)} = l_\gamma$ and up to the extraction of a subsequence, we can build $\{r_{(\gamma,i)}^\epsilon\}$ such that $r_{(\gamma,i)}^\epsilon > 0$ and $r_{(\gamma,i)}^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ with

$$\int_{I_\xi^\epsilon(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon) \cap S_\gamma^\epsilon} e^{\check{u}_\gamma^\epsilon} d\theta \rightarrow m_{(\gamma,i)} \text{ as } \epsilon \rightarrow 0.$$

We define

$$\bar{\eta}_{(\gamma,i)}^\epsilon = \eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)})^{\frac{1}{2}}, r_{(\gamma,i)}} \circ \lambda^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{gl_\gamma, x_{l_\gamma}}^{-1}$$

and

$$\bar{\eta}_{(\gamma,0)}^\epsilon = 1 - \sum_{i=1}^{N_\gamma} \eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)})^{\frac{1}{4}}, (r_{(\gamma,i)})^{\frac{1}{2}}} \circ \lambda^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{gl_\gamma, x_{l_\gamma}}^{-1}$$

naturally extended by a constant on M so that $\bar{\eta}_{(\gamma,i)}^\epsilon \in \mathcal{C}^\infty(M)$. For $0 \leq i \leq N_\gamma$ the function

$$\eta_{(\gamma,i)}^\epsilon = \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon$$

satisfies (6.67) in the proof of Claim 46 and that

$$\text{supp}(\eta_{(\gamma,i)}^\epsilon) \cap \text{supp}(\eta_{(\gamma,j)}^\epsilon) = \emptyset \text{ for } i \neq j \text{ and } \text{supp}(\eta_{(\gamma,i)}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon).$$

The use of these test functions proves that $N_\gamma \leq k$.

Let $1 \leq i \leq N_\gamma$. We define the sets

$$\begin{aligned} \Gamma_{(\gamma,i)}^\epsilon &= H_{a_{(\gamma,i)}^\epsilon, \alpha_{(\gamma,i)}^\epsilon}^{-1} \left(H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon} \left(\Gamma_\gamma^\epsilon \cap \lambda^{-1} \left(I_\zeta^\epsilon(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon) \right) \right) \right), \\ S_{(\gamma,i)}^\epsilon &= \lambda^{-1} \left(D_{(\gamma,i)}^\epsilon \right), \\ I_{(\gamma,i)}^\epsilon &= \exp_{g_{l_\gamma}, x_{l_\gamma}} \left(H_{a_{(\gamma,i)}^\epsilon, \alpha_{(\gamma,i)}^\epsilon} \left(\Gamma_{(\gamma,i)}^\epsilon \right) \right) = \check{I}_{(\gamma,i)}^\epsilon, \\ I_{(\gamma,0)}^\epsilon &= I_\gamma^\epsilon \setminus \bigcup_{i=1}^{N_\gamma} I_{(\gamma,i)}^\epsilon \end{aligned}$$

and the densities

$$\begin{aligned} \frac{e^{\hat{u}_{(\gamma,i)}^\epsilon \left(\frac{z - a_{(\gamma,i)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \right)}}{\alpha_{(\gamma,i)}^\epsilon} &= \frac{e^{\hat{u}_\gamma^\epsilon \left(\frac{z - a_\gamma^\epsilon}{\alpha_\gamma^\epsilon} \right)}}{\alpha_\gamma^\epsilon}, \\ e^{\check{u}_{(\gamma,i)}^\epsilon} ds &= \lambda^* \left(e^{\hat{u}_{(\gamma,i)}^\epsilon} d\theta \right), \end{aligned}$$

and by Hersch's normalization, we choose the parameters $a_{(\gamma,i)}^\epsilon$ and $\alpha_{(\gamma,i)}^\epsilon$ with

$$\int_{S^1} x e^{\check{u}_{(\gamma,i)}^\epsilon} \mathbf{1}_{S_{(\gamma,i)}^\epsilon} d\theta = 0 \quad (6.66)$$

and

$$\int_{I_{(\gamma,i)}^\epsilon} e^{u_\epsilon} d\sigma_g = \int_{\Gamma_{(\gamma,i)}^\epsilon} e^{\hat{u}_{(\gamma,i)}^\epsilon} ds = \int_{S_{(\gamma,i)}^\epsilon} e^{\check{u}_{(\gamma,i)}^\epsilon} d\theta = m_{(\gamma,i)}.$$

Claim 46. T is a finite tree.

Proof.

STEP 1 - We prove that if $\gamma \in T \setminus L_T$, then

$$\text{either } N_\gamma = 0 \text{ or } \#\{0 \leq i \leq N_\gamma; m_{(\gamma,i)} > 0\} \geq 2$$

Since $m_{(\gamma,i)} > 0$ for $1 \leq i \leq N_\gamma$, we get Step 1 if $N_\gamma \geq 2$ or $N_\gamma = 0$. We now assume that $N_\gamma = 1$. By (6.65) and (6.66),

$$\int_{S^1} (x, z_{(\gamma,1)}) dv_{(\gamma,0)} + m_{(\gamma,1)} = 0$$

Since $m_{(\gamma,1)} > 0$, we get that $v_{(\gamma,0)} \neq 0$ and $m_{(\gamma,0)} > 0$. This proves Step 1.

STEP 2 - We prove that if $\gamma \in T \setminus L_T$, then

$$\frac{\int_M \left| \nabla \eta_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_{\partial M} \left(\eta_{(\gamma,i)}^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \quad (6.67)$$

and that if $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}|$, then

- If $\tilde{\gamma}$ is not a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \cap \text{supp}(\eta_\gamma^\epsilon) = \emptyset$.
- If $\tilde{\gamma}$ is a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon)$.

We prove (6.67) by induction on $|\gamma|$. This is clearly true for $|\gamma| = 1$. Let $j \geq 1$ and assume that (6.67) holds for all $|\gamma| \leq j$. We have

$$\frac{\int_M \left| \nabla \eta_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_{\partial M} \left(\eta_{(\gamma,i)}^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g} = \frac{\int_M \left| \nabla \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g}{\int_{\partial M} \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g}$$

with

$$\begin{aligned} \int_M \left| \nabla \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g &\leq 2 \left(\int_M \left| \nabla \eta_\gamma^\epsilon \right|_g^2 dv_g + \int_M \left| \nabla \bar{\eta}_{(\gamma,i)}^\epsilon \right|_g^2 dv_g \right) \\ &= 2 \left(o \left(\int_{\partial M} \left(\eta_\gamma^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g \right) + o(1) \right) \end{aligned}$$

by the induction assumption, and for $i \geq 1$,

$$\begin{aligned} \int_{\partial M} \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g &\geq \int_{S^1} \left(\eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}, r_{(\gamma,i)}^\epsilon} \right)^2 e^{u_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} d\theta \\ &\geq \int_{S^1} e^{u_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon \cap I_\xi^\epsilon(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon)} d\theta \\ &= m_{(\gamma,i)} \end{aligned}$$

and for $i = 0$, fixing $\rho > 0$,

$$\begin{aligned} \int_{\partial M} \left(\eta_\gamma^\epsilon \bar{\eta}_{(\gamma,0)}^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g &\geq \int_{S^1} \left(1 - \sum_{i=1}^{N_\gamma} \eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{4}}, (r_{(\gamma,i)}^\epsilon)^{\frac{1}{2}}} \right)^2 e^{u_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} d\theta \\ &\geq \int_{S^1 \setminus \bigcup_{i=1}^{N_\gamma} I_\xi^\epsilon(p_i, \rho)} e^{u_\gamma^\epsilon} \mathbf{1}_{S_\gamma^\epsilon} d\theta \\ &= \int_{S^1 \setminus \bigcup_{i=1}^{N_\gamma} I_\xi^\epsilon(p_i, \rho)} \left(dv_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}} \right) + o(1) \\ &= \int_{S^1 \setminus I_\xi^\epsilon(p_i, \rho)} dv_{(\gamma,0)} + o(1) \end{aligned}$$

as $\epsilon \rightarrow 0$. Gathering the previous inequalities, together with

$$\int_{\partial M} \left(\eta_\gamma^\epsilon \right)^2 e^{u_\epsilon} d\sigma_g \leq \int_{\partial M} e^{u_\epsilon} d\sigma_g = 1,$$

we get (6.67).

We now prove the second part of step 2, also by induction. Assume that, for some $j \geq 1$ fixed, for all $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j$ we have that

- If $\tilde{\gamma}$ is not a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \cap \text{supp}(\eta_\gamma^\epsilon) = \emptyset$.
- If $\tilde{\gamma}$ is a descendant of γ , then $\text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\eta_\gamma^\epsilon)$.

Let us prove now that this is still true for any $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j+1$. If $|\tilde{\gamma}| \leq j$, there is of course nothing to prove. Assume that $|\tilde{\gamma}| = j+1$

If $|\gamma| = j+1$, then,

$$\text{supp}(\eta_\gamma^\epsilon) \cap \text{supp}(\eta_{\tilde{\gamma}}^\epsilon) \subset \text{supp}(\bar{\eta}_\gamma^\epsilon) \cap \text{supp}(\bar{\eta}_{\tilde{\gamma}}^\epsilon)$$

which is empty if and only if $\gamma \neq \tilde{\gamma}$.

If $|\gamma| \leq j$, we denote $\tilde{\gamma} = (\hat{\gamma}, i)$ with $0 \leq i \leq N_{\hat{\gamma}}$. We can apply the induction hypothesis to $|\gamma| \leq |\hat{\gamma}| \leq j$. Then,

- if $\text{supp}(\eta_{\tilde{\gamma}}) \cap \text{supp}(\eta_\gamma) \neq \emptyset$, we get $\text{supp}(\eta_{\hat{\gamma}}) \cap \text{supp}(\eta_\gamma) \neq \emptyset$ since $\text{supp}(\eta_{\tilde{\gamma}}) \subset \text{supp}(\eta_{\hat{\gamma}})$. By the induction assumption, $\hat{\gamma}$ is a descendant of γ and $\tilde{\gamma}$ is a descendant of γ .
- If $\tilde{\gamma}$ is a descendant of γ , then, $\hat{\gamma}$ is a descendant of γ and by the induction assumption, $\text{supp}(\eta_{\tilde{\gamma}}) \subset \text{supp}(\eta_{\hat{\gamma}}) \subset \text{supp}(\eta_\gamma)$.

The proof of Step 2 is complete.

STEP 3 - We prove the following assertion \mathbf{H}_j by induction on j .

\mathbf{H}_j : If $T_j \neq T_{j+1}$, then, $T_{j+1} = T$ or there exist $j+1$ test functions with pairwise disjoint support in the set $\{\eta_\gamma^\epsilon, \gamma \in T_{j+1}\}$.

Notice that by (6.67) in Step 2, the assumption $T_{k+1} \neq T$ would give a contradiction. Indeed, it suffices to test the $k+1$ functions given by the assumption \mathbf{H}_{k+1} in the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$, (6.4). Therefore, the increasing sequence of trees $\{T_j\}$ is stationnary, and Claim 46 will follow.

Note that \mathbf{H}_1 is true by the existence of $\{\eta_1^\epsilon\}$.

Let $j \geq 2$ and we assume that \mathbf{H}_{j-1} is true and that $T_j \neq T_{j+1}$. Then, $T_{j-1} \neq T_j$ and \mathbf{H}_{j-1} gives j test functions with pairwise disjoint support in the set $\{\eta_\gamma^\epsilon; \gamma \in T_j\}$ denoted by $\eta_{\gamma_1}^\epsilon \cdots \eta_{\gamma_j}^\epsilon$. We assume that $T_{j+1} \neq T$. Then, there is $\gamma \in T_j$ such that $N_\gamma \geq 1$. By Step 1, there are two indices $i_1 \neq i_2$ such that $m_{(\gamma, i_1)} > 0$ and $m_{(\gamma, i_2)} > 0$.

If γ is not a descendant of one of $\gamma_1, \dots, \gamma_j$, then we take the set of test functions

$$\{\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon, \eta_{(\gamma, i_1)}^\epsilon\}.$$

If γ is a descendant of one of $\gamma_1, \dots, \gamma_j$, then, by Step 2, since the functions $\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon$ have pairwise disjoint support, there is a unique $1 \leq i \leq j$ such that γ is a descendant of γ_i and we take the set of test functions with pairwise disjoint support

$$\{\eta_{\gamma_1}^\epsilon, \dots, \eta_{\gamma_{i-1}}^\epsilon, \eta_{\gamma_{i+1}}^\epsilon, \dots, \eta_{\gamma_j}^\epsilon, \eta_{(\gamma, i_1)}^\epsilon, \eta_{(\gamma, i_2)}^\epsilon\}.$$

Thus \mathbf{H}_j holds. This ends the proof of Step 3 and as already said the proof of the claim. \diamond

Thanks to this construction, the parameters $(a_\gamma^\epsilon, \alpha_\gamma^\epsilon)$ define separated bubble or bubbles over bubbles. This reads as a formula which originates from [8] and [106] in the context of bubble tree constructions :

Claim 47. If $\gamma \in T \setminus L_T$, $\alpha_\gamma^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and if $\gamma_1, \gamma_2 \in T \setminus L_T$ with $\gamma_1 \neq \gamma_2$, then

$$\frac{d_g(\bar{a}_{\gamma_1}^\epsilon, \bar{a}_{\gamma_2}^\epsilon)}{\alpha_{\gamma_1}^\epsilon + \alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_1}^\epsilon}{\alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_2}^\epsilon}{\alpha_{\gamma_1}^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Proof. We recall that there exists $C_0 > 0$ such that for all $0 < r < \delta$,

$$B_g(x, C_0^{-1}r) \subset \exp_{g_l, x_l}(\mathbb{D}_r^+(\hat{x}^l)) \subset B_g(x, C_0 r)$$

for all $x \in \gamma_l$ with $1 \leq l \leq L$. On the discs, there also exists $C_1 > 0$ and some $\delta_1 > 0$ such that for all $0 < r < \delta_1$,

$$B_{\tilde{g}}(z_\gamma, C_1^{-1}r) \subset \lambda^{-1}(\mathbb{D}_r^+(\hat{z}_\gamma)) \subset B_{\tilde{g}}(z_\gamma, C_1 r)$$

for all $\gamma \in T \setminus L_T$ such that $|\gamma| \geq 2$ and $z_\gamma \neq p$, where $\hat{z}_\gamma = \lambda(z_\gamma)$; and

$$B_{\tilde{g}}(p, C_1^{-1}r) \subset \lambda^{-1}(\mathbb{R}_+^2 \setminus \mathbb{D}_{\frac{1}{r}}^+) \subset B_{\tilde{g}}(p, C_1 r).$$

Now, given $\gamma_1, \gamma_2 \in T \setminus L_T$, we let $\gamma \in T$ such that $\gamma_1 = (\gamma, \tilde{\gamma}_1)$, $\gamma_2 = (\gamma, \tilde{\gamma}_2)$ and $|\gamma|$ is maximal. We consider 5 cases in order to prove the claim.

CASE 1 - $\gamma = (\emptyset)$. Then $\gamma_1 = (i, \hat{\gamma}_1)$ and $\gamma_2 = (j, \hat{\gamma}_1)$ with $i \neq j$.

Since

$$I_{\gamma_1}^\epsilon \subset I_g(z_i, r_i^\epsilon) \subset \exp_{g_l, x_l}(I_{C_0 r_i^\epsilon}(\tilde{z}_i)),$$

we get with (6.64) that

$$|a_i^\epsilon - \tilde{z}_i| \leq C_0 r_i^\epsilon$$

and

$$\alpha_i^\epsilon \leq C_0 r_i^\epsilon + |a_i^\epsilon - \tilde{z}_i|$$

so that $a_i^\epsilon \rightarrow \tilde{z}_i$ as $\epsilon \rightarrow 0$ and $\alpha_i^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and the same is true for j . Then, since $z_i \neq z_j$,

$$\frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon + \alpha_j^\epsilon} = \frac{d_g(z_i, z_j) + o(1)}{\alpha_i^\epsilon + \alpha_j^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

CASE 2 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma, j)} \neq p$.

Then, we have

$$I_{\gamma_2}^\epsilon \subset I_{(\gamma, j)}^\epsilon \subset \exp_{g_l, x_l}(I_{C_1 r_{(\gamma, j)}^\epsilon}^{\alpha_\gamma^\epsilon}(\alpha_\gamma^\epsilon \hat{z}_{(\gamma, j)} + a_\gamma^\epsilon))$$

so that by (6.66), we have that

$$|\alpha_\gamma^\epsilon \hat{z}_{(\gamma, j)} + a_\gamma^\epsilon - a_{(\gamma, j)}^\epsilon| \leq C_1 r_{(\gamma, j)}^\epsilon \alpha_\gamma^\epsilon$$

and

$$\alpha_{(\gamma, j)}^\epsilon \leq C_1 r_{(\gamma, j)}^\epsilon \alpha_\gamma^\epsilon + |\alpha_\gamma^\epsilon \hat{z}_{(\gamma, j)} + a_\gamma^\epsilon - a_{(\gamma, j)}^\epsilon|$$

and $\frac{\alpha_\gamma^\epsilon}{\alpha_{(\gamma, j)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

CASE 3 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma, j)} = p$.

We assume that $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_{(\gamma,j)}^\epsilon + \alpha_\gamma^\epsilon}$ is bounded and we prove by contradiction that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. We assume that $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_\gamma^\epsilon)$. Then, it is clear that $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon}$ is bounded and we have by (6.66) that

$$\alpha_{(\gamma,j)}^\epsilon \geq \frac{\alpha_\gamma^\epsilon}{C_1 r_{(\gamma,i)}^\epsilon} - |a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon|$$

so that

$$\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \geq \frac{1}{C_1 r_{(\gamma,i)}^\epsilon} - \frac{|a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon|}{\alpha_\gamma^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

which contradicts the assumption $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_\gamma^\epsilon)$. Thus, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

CASE 4 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $i \neq j$, $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} \neq p$.

We have that $|a_{(\gamma,i)}^\epsilon - a_{(\gamma,j)}^\epsilon| = \alpha_\gamma^\epsilon \left(|\hat{z}_{(\gamma,i)} - \hat{z}_{(\gamma,j)}| + o(1) \right)$, $\frac{\alpha_{(\gamma,i)}^\epsilon}{\alpha_\gamma^\epsilon} = o(1)$ and $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} = o(1)$ by Case 2 so that

$$\frac{d_g(\bar{a}_{(\gamma,i)}^\epsilon, \bar{a}_{(\gamma,j)}^\epsilon)}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

CASE 5 - $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} = p$.

As in Case 3, we assume that $\frac{|a_{(\gamma,i)}^\epsilon - a_{(\gamma,j)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon}$ is bounded and we will prove by contradiction that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. Let's assume that $\alpha_{(\gamma,j)}^\epsilon = O(\alpha_{(\gamma,i)}^\epsilon)$. Then,

$$\begin{aligned} \frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + \frac{|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon} \\ &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + \frac{|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + o(\alpha_\gamma^\epsilon)} \\ &\leq \frac{|a_{(\gamma,j)}^\epsilon - a_{(\gamma,i)}^\epsilon|}{\alpha_{(\gamma,i)}^\epsilon + \alpha_{(\gamma,j)}^\epsilon} + O(1) \end{aligned}$$

since $\alpha_{(\gamma,i)}^\epsilon = o(\alpha_\gamma^\epsilon)$ by Case 2, and $|a_{(\gamma,i)}^\epsilon - a_\gamma^\epsilon| = O(\alpha_\gamma^\epsilon)$. Then, $\frac{|a_{(\gamma,j)}^\epsilon - a_\gamma^\epsilon|}{\alpha_\gamma^\epsilon + \alpha_{(\gamma,j)}^\epsilon}$ is bounded and by Case 3, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_\gamma^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ so that $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ which gives a contradiction. Thus, $\frac{\alpha_{(\gamma,j)}^\epsilon}{\alpha_{(\gamma,i)}^\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

Gathering all the cases, the proof is complete. \diamond

Now, we are in position to prove Proposition 7. We denote by $L^+ \subset L_T$ the set of leaves $\gamma \in L_T$ such that $m_\gamma > 0$.

To simplify, we now denote the elements of L^+ by $\{1, \dots, N\}$ and all the indices $\gamma \in L^+$ in I_γ^ϵ , Γ_γ^ϵ , S_γ^ϵ , a_γ^ϵ , α_γ^ϵ , $e_{\tilde{\gamma}}^\epsilon$, $e_{\tilde{\gamma}}^\epsilon$, v_γ and m_γ are replaced by the corresponding index $i \in \{1, \dots, N\}$.

Up to the extraction of a subsequence and up to reorder the α_i^ϵ 's, we get (6.56), (6.57) and (6.58) thanks to Claim 47. By construction, we obtain the remaining facts of the proposition.

6.6 Regularity estimates at the concentration scales

In this section, we aim at proving some energy estimates in order to prove later Proposition 8 page 244. We fix $i \in \{1, \dots, N\}$ given by Proposition 7 and up to the end of the section drop the index i of the parameters $l_i, a_i^\epsilon, \alpha_i^\epsilon$ the functions \hat{u}_i^ϵ , we defined. As described in Section 6.2.1, we let

$$\hat{\Phi}_\epsilon(z) = \tilde{\Phi}_\epsilon^l \circ H_{a_\epsilon, \alpha_\epsilon}(z) = \tilde{\Phi}_\epsilon^l(\alpha_\epsilon z + a_\epsilon)$$

and

$$\hat{v}_\epsilon = H_{a_\epsilon, \alpha_\epsilon}^*(\tilde{v}_\epsilon).$$

Then, for $0 \leq i \leq n$ and for $\rho > 0$ fixed, we get the equations

$$\begin{cases} \Delta_{\tilde{\zeta}} \hat{\phi}_\epsilon^i = 0 & \text{in } \mathbb{D}_{\frac{1}{\rho}}^+ \\ \partial_t \hat{\phi}_\epsilon^i = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\phi}_\epsilon^i & \text{on } I_{\frac{1}{\rho}}. \end{cases} \quad (6.68)$$

As we will see, the properties gathered in Proposition 6 and Claim 42 are in some sense invariant by dilatation. Indeed, this is clear in the equation (6.68). We also have that if $\Omega \subset \omega_l$ and $\Gamma = \Omega \cap \partial M$,

$$\sigma_*(\Omega, g, \Gamma, e^{u_\epsilon}) = \sigma_*(\hat{\Omega}, \tilde{\zeta}, \hat{\Gamma}, e^{\hat{u}_\epsilon})$$

where we set $\hat{\Omega} = H_{a_\epsilon, \alpha_\epsilon}^{-1}(\tilde{\Omega}^l)$ and $\hat{\Gamma} = H_{a_\epsilon, \alpha_\epsilon}^{-1}(\tilde{\Gamma}^l)$. The heat equation is also invariant by dilatation, up to some errors on the surface M we precised in Section 6.2.2 (see (6.5) and (6.7)), thanks to the following identity in the Euclidean case

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\epsilon}} e^{-\frac{|x-y|^2}{4\epsilon}} f(y) dy = \int_{\mathbb{R}} \frac{\alpha}{\sqrt{4\pi\epsilon}} e^{-\alpha^2 \frac{|\frac{x}{\alpha}-y|^2}{4\epsilon}} f(\alpha y) dy.$$

Therefore, we can derive regularity estimates of the eigenfunctions at all the concentration scales.

However, we have to distinguish two cases, depending on the speed of concentration α_ϵ when compared to ϵ . In section 6.6.1, we treat the case when $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$, and in section 6.6.2, we will treat the case when $\alpha_\epsilon^2 = O(\epsilon)$.

6.6.1 Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$

We assume in this subsection that $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$. We set $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)} \alpha_\epsilon^2}$, where $a_\epsilon \rightarrow a \in \mathbb{R} \times \{0\}$ as $\epsilon \rightarrow 0$, and $i_0 \in \{1, \dots, N_0\}$ such that $\tilde{z}_{i_0} = a$. Then

$$\theta_\epsilon \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (6.69)$$

We will adapt the technics of Section 6.4.1 in the surface $(\mathbb{D}^2, \tilde{\zeta}, \mathbb{S}^1, e^{\hat{u}_\epsilon})$. First, notice that

$$e^{\hat{u}_\epsilon} ds - d\hat{v}_\epsilon \rightarrow_* 0 \text{ in } \mathcal{M}(\mathbb{R} \times \{0\}) \text{ as } \epsilon \rightarrow 0. \quad (6.70)$$

Indeed, for $\zeta \in \mathcal{C}_c^0(I_{R_0})$ for some $R_0 > 0$, and $R > R_0$, we can write that

$$\begin{aligned} \int_{\mathbb{R} \times \{0\}} \zeta(x) \left(e^{2\hat{u}_\epsilon(x)} dx - d\hat{v}_\epsilon(x) \right) &= \int_{\partial M \setminus \check{I}_R} \left(\int_{\check{I}_{R_0}} p_\epsilon(y, x) \zeta(y) d\sigma_g(y) \right) dv_\epsilon(x) \\ &+ \int_{I_R} \left(\int_{I_R} (\zeta(z) - \zeta(x)) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\ &+ \int_{I_{R_0}} \left(\int_{I_R} \hat{p}_\epsilon(z, x) dz - 1 \right) \zeta(x) d\hat{v}_\epsilon(x). \end{aligned}$$

By estimates (6.10) on the heat kernel, we have that

$$\begin{aligned} \int_{\partial M \setminus \check{I}_R} \left(\int_{\check{I}_{R_0}} p_\epsilon(x, y) |\zeta(y)| d\sigma_g(y) \right) dv_\epsilon(x) &\leq \|\zeta\|_\infty \sup_{x \in \partial M \setminus \check{I}_R} \int_{\check{I}_{R_0}} p_\epsilon(x, y) d\sigma_g(y) \\ &\leq O \left(\frac{e^{-\frac{(R-R_0)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}} \right) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

By estimates (6.8) on the heat kernel, we have that

$$\begin{aligned} \int_{I_R} \left(\int_{I_R} |\zeta(z) - \zeta(x)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) &\leq \sup_{x \in I_R} \int_{\mathbb{R} \times \{0\}} |\zeta(x) - \zeta(z)| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}} dz \\ &\rightarrow 0 \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

since ζ is uniformly continuous on $\mathbb{R} \times \{0\}$. Finally, we have by the heat kernel estimate (6.11) that

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \left| \int_{I_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0,$$

so that we get (6.70). We denote by \hat{v} the weak star limit of both $\{e^{\hat{u}_\epsilon} dx\}$ and $\{\hat{v}_\epsilon\}$ in $\mathcal{M}(\mathbb{R} \times \{0\})$.

Let's tackle a generalization of Claim 42 at all the scales which appear between α_ϵ and δ_0 . For a sequence $\{\gamma_\epsilon\}$, we let

$$e^{\overline{u}_\epsilon^{\gamma_\epsilon}(x)} = \gamma_\epsilon e^{\tilde{u}_\epsilon^l(\gamma_\epsilon x + a_\epsilon)} \text{ and } \overline{\Phi}_\epsilon^{\gamma_\epsilon}(x) = \widetilde{\Phi}_\epsilon^l(\gamma_\epsilon x + a_\epsilon),$$

and for a sequence of domains $\Omega_\epsilon \subset \omega_l$, with $\Gamma_\epsilon = \partial\Omega_\epsilon \cap \partial M \neq \emptyset$.

$$\overline{\Omega}_\epsilon^{\gamma_\epsilon} = H_{a_\epsilon, \gamma_\epsilon}^{-1} \left(\tilde{\Omega}_\epsilon^l \right) \text{ and } \overline{\Gamma}_\epsilon^{\gamma_\epsilon} = H_{a_\epsilon, \gamma_\epsilon}^{-1} \left(\tilde{\Gamma}_\epsilon^l \right)$$

so that

$$\sigma_\star(\Omega_\epsilon, g, \Gamma_\epsilon, e^{\hat{u}_\epsilon}) = \sigma_\star(\overline{\Omega}_\epsilon^{\gamma_\epsilon}, \zeta, \overline{\Gamma}_\epsilon^{\gamma_\epsilon}, e^{\overline{u}_\epsilon^{\gamma_\epsilon}})$$

and

$$\begin{cases} \Delta_{\tilde{\zeta}} \overline{\Phi}_\epsilon^{\gamma_\epsilon} = 0 & \text{in } \overline{\Omega}_\epsilon^{\gamma_\epsilon} \\ \partial_t \overline{\Phi}_\epsilon^{\gamma_\epsilon} = -\sigma_\epsilon e^{\overline{u}_\epsilon^{\gamma_\epsilon}} \overline{\Phi}_\epsilon^{\gamma_\epsilon} & \text{on } \overline{\Gamma}_\epsilon^{\gamma_\epsilon}. \end{cases}$$

We also let \mathbb{A}_ρ be the half-annulus $\mathbb{D}_{\frac{1}{\rho}}^+ \setminus \mathbb{D}_\rho^+$ and $J_\rho = I_{\frac{1}{\rho}} \setminus I_\rho$.

We recall that $X_r(\Omega, \zeta, \Gamma, \{e^{\overline{u}_\epsilon^{\gamma_\epsilon}}\})$ is the set of points x of $\Omega \subset \mathbb{R}_+^2$ (with $\Gamma = \Omega \cap \mathbb{R} \times \{0\}$) such that there exists $\epsilon > 0$ which satisfies $\mathbf{P}_{r,\epsilon}$, that is $\mathbf{A}_{r,\epsilon}$ or $\mathbf{B}_{r,\epsilon}$, where

$$\mathbf{A}_{r,\epsilon} : x \in \Gamma \text{ and } \sigma_*(\mathbb{D}_r(x), \zeta, I_r(x), e^{\overline{u}_\epsilon^{\gamma_\epsilon}}) \leq \frac{\sigma_k(M, [g])}{2}$$

$$\mathbf{B}_{r,\epsilon} : \text{There exists } f \in E_k(M, g, \partial M, e^{u_\epsilon}) \text{ such that } \bar{f}^{\gamma_\epsilon}(x) = 0 \text{ and the Nodal set of } \bar{f}^{\gamma_\epsilon} \text{ which contains } x \text{ does not intersect } \partial \mathbb{D}_r^+(x).$$

Note that for $\gamma_\epsilon = \alpha_\epsilon$, $e^{\overline{u}_\epsilon^{\gamma_\epsilon}} = e^{\hat{u}_\epsilon}$ and that the set of concentration points satisfies

$$Z(\Omega, \{e^{\hat{u}_\epsilon} ds\}) \subset X_r(\Omega, \zeta, \Gamma, \{e^{\hat{u}_\epsilon}\}) \quad (6.71)$$

for all $r > 0$. We write $\omega_1^\epsilon \ll \omega_2^\epsilon$ if two sequences $\{\omega_1^\epsilon\}$ and $\{\omega_2^\epsilon\}$ satisfy $\frac{\omega_1^\epsilon}{\omega_2^\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Claim 48. *Up to the extraction of a subsequence, there exist some sequences $\{\omega_i^\epsilon\}$ with $0 \leq i \leq t+1$ and $0 \leq t \leq k$ such that*

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \dots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0 ,$$

there exist $R_0 > 0$ and some points $p_{i,j}$ with $0 \leq i \leq t$ and $1 \leq j \leq s_i$ such that if $1 \leq i \leq t$, $p_{i,j} \in J_{\frac{1}{R_0}}$ and if $i = 0$, $p_{0,j} \in I_{R_0}$, with

$$s - 1 + \sum_{i=0}^t s_i \leq k$$

and for all $0 < \rho < \frac{1}{2R_0}$, there exists some $r > 0$ such that for all $1 \leq i \leq t$,

$$X_r(\mathbb{A}_\rho, \zeta, J_\rho, \{e^{\overline{u}_\epsilon^{\omega_i^\epsilon}}\}) \subset \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}) ,$$

$$X_r(\mathbb{D}_{\frac{1}{\rho}}^+, \zeta, I_{\frac{1}{\rho}}, \{e^{\hat{u}_\epsilon}\}) \subset \bigcup_{j=1}^{s_0} \mathbb{D}_\rho^+(p_{0,j}) ,$$

for all sequence $\{\gamma_\epsilon\}$ such that $\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho \omega_{i+1}^\epsilon$ with $0 \leq i \leq t$ fixed,

$$X_r(\mathbb{A}_{R_0\rho}, \zeta, J_{R_0\rho}, \{e^{\overline{u}_\epsilon^{\gamma_\epsilon}}\}) = \emptyset ,$$

and for all $0 < \rho < \frac{1}{2R_0}$, for all $r > 0$, for all $0 \leq i \leq t$, $1 \leq j \leq s_i$ and for all subsequence $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$,

$$X_r\left(\mathbb{D}_{\frac{1}{\rho}}^+, \zeta, I_{\frac{1}{\rho}}, \{e^{\overline{u}_{\epsilon_m}^{\omega_i^{\epsilon_m}}}\}_{m \geq 0}\right) \cap \mathbb{D}_\rho^+(p_{i,j}) \neq \emptyset . \quad (6.72)$$

Proof.

By contradiction, we assume that for all subsequence $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$, for all $\{\omega_i^{\epsilon_m}\}_{m \geq 0}$ with $0 \leq i \leq t$ and

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \dots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0 ,$$

for all families of points $p_{i,j} \in \mathbb{R}^2$ with $0 \leq i \leq t$ and $1 \leq j \leq s_i$ such that if $1 \leq i \leq t$, $p_{i,j} \in J_{\frac{1}{R_0}}$ and if $i = 0$, $p_{0,j} \in I_{R_0}$, with

$$s - 1 + \sum_{i=0}^t s_i \leq k$$

and

$$R_0 = \max \left\{ \max_{1 \leq i \leq t, 1 \leq j \leq s_i} \left\{ \max \left\{ |p_{i,j}|, \frac{1}{|p_{i,j}|} \right\} \right\}, \max_{1 \leq j \leq s_0} \{|p_{0,j}|\}, \delta_0 \right\} + 1,$$

there exists $0 < \rho < \frac{1}{2R_0}$ such that for all $r > 0$, either there exists $1 \leq i \leq t$ such that

$$X_r \left(\mathbb{A}_\rho, \zeta, J_\rho, \{e^{\overline{u}_\epsilon^\epsilon}\} \right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}) \neq \emptyset, \quad (6.73)$$

or

$$X_r \left(\mathbb{D}_{\frac{1}{\rho}}^+, \zeta, I_{\frac{1}{\rho}}, \{e^{\hat{u}_\epsilon}\} \right) \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_\rho^+(p_{0,j}) \neq \emptyset, \quad (6.74)$$

or there exists a sequence $\{\gamma_\epsilon\}$ such that $\frac{\omega_\epsilon}{\rho} < \gamma_\epsilon < \rho\omega_{i+1}^\epsilon$ for some $0 \leq i \leq t$, with

$$X_r \left(\mathbb{A}_{R_0\rho}, \zeta, J_{R_0\rho}, \{e^{\overline{u}_\epsilon^{\gamma_\epsilon}}\} \right) \neq \emptyset. \quad (6.75)$$

With this assumption, we prove by induction the following property $\mathbf{H}_{\tilde{s}}$ for $s - 1 \leq \tilde{s} \leq k + 1$

$\mathbf{H}_{\tilde{s}}$: there exist sequences $\epsilon_m \rightarrow 0$ and $r_m \searrow 0$ as $m \rightarrow +\infty$, some scales

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \cdots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0,$$

some points $p_{i,j}^m \in \mathbb{R}_+^2 \setminus \{0\}$ and $p_{i,j} \in \mathbb{R} \times \{0\}$ if $1 \leq i \leq t$, $1 \leq j \leq s_i$; and $p_{0,j}^m \in \mathbb{R}^2$ and $p_{0,j} \in \mathbb{R} \times \{0\}$ if $1 \leq j \leq s_0$ with

$$s - 1 + \sum_{i=0}^t s_i = \tilde{s}$$

and $p_{i,j} \neq p_{i,j'}$ if $j \neq j'$ for $0 \leq i \leq t$, such that for all $0 \leq i \leq t$, $1 \leq j \leq s_i$, $p_{i,j}^m$ satisfies $\mathbf{P}_{r_m, \epsilon_m}$ in $\left(\mathbb{R}_+^2, \zeta, \mathbb{R} \times \{0\}, \{e^{\overline{u}_{\epsilon_m}^{\omega_i^{\epsilon_m}}}\} \right)_{m \geq 0}$.

We already have \mathbf{H}_{s-1} , let's prove \mathbf{H}_s . We fix $\rho > 0$. By assumption, since we apply it with all s_i 's equal to 0, either (6.75) or (6.74) happen. Let's study these two cases :

CASE (6.75) $_{s-1}$: There exists a sequence $\{\gamma_{\epsilon_m}\}$ with $\frac{\alpha_{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho\delta_0$ and some $x_m \in X_{2^{-m}} \left(\mathbb{A}_\rho, \zeta, J_\rho, \{e^{\overline{u}_\epsilon^{\gamma_\epsilon}}\} \right)$. We choose ϵ_m such that x_m satisfies $\mathbf{P}_{2^{-m}, \epsilon_m}$. It is clear that $\epsilon_m \rightarrow 0$ as $m \rightarrow \infty$.

— If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}} \rightarrow +\infty$, we set a new scale $\omega_1^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{1,1}^m = x_m \in \mathbb{A}_\rho$. Up to the extraction of a subsequence, $p_{1,1}^m \rightarrow p_{1,1} \in \mathbb{R}_+^2 \setminus \{0\}$ as $m \rightarrow +\infty$. It is clear by Claim 42 that $\omega_1^{\epsilon_m} \ll \delta_0$ up to reduce ρ . By the same arguments as in Claim 42, $p_{1,1} \in \mathbb{R} \times \{0\} \setminus \{0\}$ and we get \mathbf{H}_s in this case.

- If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}}$ is bounded, up to reduce ρ , one gets that (6.74) holds and we can go to Case (6.74)_{s-1}.

CASE (6.74)_{s-1} : There exists $x_m \in X_{2-m} \left(\mathbb{D}_{\frac{1}{\rho}}^+, \xi, I_{\frac{1}{\rho}} \{e^{\hat{u}_\epsilon}\} \right)$. We set $p_{0,1}^m = x_m$ and up to the extraction of a subsequence, $p_{0,1}^m \rightarrow p_{0,1}$ as $m \rightarrow +\infty$. By the same arguments as in Claim 42, $p_{0,1} \in \mathbb{R} \times \{0\}$ and we get \mathbf{H}_s in this case.

Now, we assume that $\mathbf{H}_{\tilde{s}}$ is true for some $s \leq \tilde{s} \leq k$. Let's prove $\mathbf{H}_{\tilde{s}+1}$. We define all the parameters $\epsilon_m, r_m, \omega_i^{\epsilon_m}, p_{i,j}^m$ and $p_{i,j}$ given by $\mathbf{H}_{\tilde{s}}$. We fix $\rho > 0$. By assumption, one of the assertions (6.73), (6.74) and (6.75) must happen. Let's study these three cases :

CASE (6.73) _{\tilde{s}} : Let $1 \leq i \leq t$ and $x_m \in X_{r_m} \left(\mathbb{A}_\rho, \xi, J_\rho, \{e^{\overline{u_{\epsilon_m} \omega_i^{\epsilon_m}}}\} \right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j})$. For $m \geq 0$, we set $p_{i,s_i+1}^m = x_m$ and we let $\epsilon_{\beta(m)}$ be such that p_{i,s_i+1}^m satisfies $\mathbf{P}_{r_m, \epsilon_{\beta(m)}}$. Since $r_m \searrow 0$, as $m \rightarrow +\infty$, setting $M(m) = \min\{m, \beta(m)\}$ gives that $p_{i,j}^{\beta(m)}$ satisfies $\mathbf{P}_{r_{M(m)}, \epsilon_{\beta(m)}}$ for all $1 \leq j \leq s_i$ and p_{i,s_i+1}^m satisfies $\mathbf{P}_{r_{M(m)}, \epsilon_{\beta(m)}}$. Up to the extraction of a subsequence, we can assume that $p_{i,s_i+1}^m \rightarrow p_{i,s_i+1}$ as $m \rightarrow +\infty$ and that $r_{M(m)} \searrow 0$ as $m \rightarrow +\infty$. Since $p_{i,s_i+1}^m \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j})$, $p_{i,s_i+1} \in \mathbb{R}_+^2 \setminus \{0, p_{i,1}, \dots, p_{i,s_i}\}$. By the same arguments as in Claim 42, $p_{i,s_i+1} \in \mathbb{R} \times \{0\} \setminus \{0\}$ and the proof of $\mathbf{H}_{\tilde{s}+1}$ is complete in this case.

CASE (6.74) _{\tilde{s}} : The proof of $\mathbf{H}_{\tilde{s}+1}$ is the same as in (6.73) _{\tilde{s}} .

CASE (6.75) _{\tilde{s}} : Let $\{\gamma_{\epsilon_m}\}$ be a sequence such that

$$\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho \omega_{i+1}^{\epsilon_m} \text{ and } x_m \in X_{r_m} \left(\mathbb{A}_{R_0 \rho}, \xi, J_{R_0 \rho}, \{e^{\overline{u_{\epsilon_q} \gamma_{\epsilon_q}}}\}_{q \geq 0} \right).$$

- If $\frac{\gamma_{\epsilon_m}}{\omega_i^{\epsilon_m}} \rightarrow +\infty$ and $\frac{\gamma_{\epsilon_m}}{\omega_{i+1}^{\epsilon_m}} \rightarrow 0$, we define a new scale $\omega_{t+1}^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{t+1,1}^m = x_m$. Up to the extraction of a subsequence, $p_{t+1,1}^m \in \mathbb{A}_\rho$ satisfies $\mathbf{P}_{r_m, \epsilon_m}$, $p_{t+1,1}^m \rightarrow p_{t+1,1} \in \mathbb{R}_+^2 \setminus \{0\}$ and $r_m \searrow 0$ as $m \rightarrow +\infty$. By the same arguments as in Claim (42), $p_{t+1,1} \in \mathbb{R} \times \{0\} \setminus \{0\}$. Up to reorder $\{\omega_i^{\epsilon_m}\}$, we get $\mathbf{H}_{\tilde{s}+1}$ in this case.
- If $i = 0$ and $\frac{\gamma_{\epsilon_m}}{\omega_0^{\epsilon_m}}$ is bounded, up to reduce ρ , we get that (6.74) holds and go back to Case (6.74) _{\tilde{s}} .
- The case $i = t$ and $\frac{\omega_{t+1}^{\epsilon_m}}{\gamma_{\epsilon_m}}$ is bounded leads to a contradiction by Claim 42.
- The other cases lead to the fact that (6.73) holds up to reduce ρ and we are back to Case (6.73) _{\tilde{s}} .

Gathering the three cases, we deduce $\mathbf{H}_{\tilde{s}+1}$. Therefore, \mathbf{H}_{k+1} holds true and we now prove that this leads to a contradiction. We will define new test functions for the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$, $\eta_{i,j}^m$ for $0 \leq i \leq t, 1 \leq j \leq s_i$.

- If $p_{i,j}^m$ satisfies $\mathbf{A}_{r_m, \epsilon_m}$, $\eta_{i,j}^m$ is defined by the extension by 0 in $M \setminus \Omega_{i,j}^m$ of an eigenfunction for $\sigma_\star \left(\Omega_{i,j}^m, g, \Gamma_{i,j}^m, e^{u_{\epsilon_m}} \right)$, where $\Omega_{i,j}^m \subset M$ and $\Gamma_{i,j}^m \subset \partial M$ are defined by $\mathbb{D}_{r_m}^+(p_{i,j}^m) = \overline{\Omega_{i,j}^m}^{\omega_i^{\epsilon_m}}$ and $I_{r_m}(p_{i,j}^m) = \overline{\Gamma_{i,j}^m}^{\omega_i^{\epsilon_m}}$.
- If $p_{i,j}^m$ does not satisfy $\mathbf{A}_{r_m, \epsilon_m}$, it satisfies $\mathbf{B}_{r_m, \epsilon_m}$ and $\eta_{i,j}^m$ is defined by an eigenfunction for $\sigma_\star \left(D_{i,j}^m, g, \Gamma_{i,j}^m, e^{u_{\epsilon_m}} \right)$ extended by 0 in $M \setminus D_{i,j}^m$, where $D_{i,j}^m \subset M$ is the Nodal domain of an eigenfunction associated to σ_{ϵ_m} such that $\overline{D_{i,j}^m}^{\omega_i^{\epsilon_m}} \subset \mathbb{D}_{r_m}^+(p_{i,j}^m)$ and $\Gamma_{i,j}^m = D_{i,j}^m \cap \partial M$.

We also use the functions η_i^m for $\{1 \leq i \leq s\}$, already defined in the proof of Claim 42. Note that these $k+1$ functions have pairwise disjoint support for m large enough. Then, by (6.4),

$$\sigma_{\epsilon_m} \leq \max \left\{ \max_{0 \leq i \leq t, 1 \leq j \leq s_i} \frac{\int_M |\nabla \eta_{i,j}^m|_g^2 dv_g}{\int_{\partial M} (\eta_{i,j}^m)^2 e^{u_{\epsilon_m}} d\sigma_g}, \max_{i \neq i_0} \frac{\int_M |\nabla \eta_i^m|_g^2 dv_g}{\int_{\partial M} (\eta_i^m)^2 e^{u_{\epsilon_m}} d\sigma_g} \right\} \leq \sigma_{\epsilon_m}.$$

The last inequality comes from the definition of the properties **A** and **B** and we have equality if and only if one of the test functions is an eigenfunction for $\sigma_{\epsilon_m} = \sigma_k(M, g, \partial M, e^{u_{\epsilon_m}})$. This test function is a non-zero harmonic function which vanishes on an open set of the surface. This is absurd.

Therefore we proved the first part of the claim. Up to make successive extractions of subsequences of $\{\epsilon_m\}$ and up to remove some points $p_{i,j}$, one easily proves that the last condition (6.72) also holds.

◇

For $\rho > 0$, we set

$$\Omega(\rho) = \mathbb{D}_{\frac{1}{\rho}}^+ \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_{\rho}^+(p_{0,j}) \text{ and } \Gamma(\rho) = I_{\frac{1}{\rho}} \setminus \bigcup_{j=1}^{s_0} I_{\rho}(p_{0,j}).$$

As previously remarked, the set of concentration points of $\{e^{\hat{u}_{\epsilon}} ds\}$ satisfies

$$Z(\mathbb{R} \times \{0\}, \{e^{\hat{u}_{\epsilon}} dx\}) \subset \{p_{0,1}, \dots, p_{0,s_0}\} \quad (6.76)$$

and letting

$$m_i(\rho) = \lim_{\epsilon \rightarrow 0} \int_{\Gamma(\rho)} e^{\hat{u}_{\epsilon}} ds,$$

we have that $m_i(\rho) \geq m_i + o(1) > 0$ since we have (6.59), (6.61), (6.76) and $m_i > 0$. We aim at getting regularity estimates on $\hat{\Phi}_{\epsilon}$ and $e^{\hat{u}_{\epsilon}}$ in $\Omega(\rho)$. We follow the proof of Claim 43, thanks to the fact that $m_i(\rho) > 0$ for ρ small enough.

Claim 49. *We have the following*

— *Estimates on $\hat{\Phi}_{\epsilon}$:*

$$\forall \rho > 0, \exists C_1(\rho) > 0, \forall \epsilon > 0, \|\hat{\Phi}_{\epsilon}\|_{W^{1,2}(\Omega(\rho))} \leq C_1(\rho), \quad (6.77)$$

$$\forall \rho > 0, \exists C_2(\rho) > 0, \forall \epsilon > 0, \|\hat{\Phi}_{\epsilon}\|_{C^0(\Omega(\rho))} \leq C_2(\rho). \quad (6.78)$$

— *Quantitative non-concentration estimates on $e^{2\hat{u}_{\epsilon}}$ and $|\nabla \hat{\Phi}_{\epsilon}|^2$*

$$\forall \rho > 0, \exists D_1(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in \Gamma(\rho)} \int_{I_r(x)} e^{\hat{u}_{\epsilon}} \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})}, \quad (6.79)$$

$$\forall \rho > 0, \exists D_2(\rho) > 0, \forall r > 0, \limsup_{\epsilon \rightarrow 0} \sup_{x \in \Gamma(\rho)} \int_{\mathbb{D}_r^+(x)} |\nabla \hat{\Phi}_{\epsilon}|^2 \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}}. \quad (6.80)$$

Proof.

The proof of (6.77) follows exactly the proof of (6.31) in Claim 43, using the fact that $m_0^i(\rho) > 0$ for ρ small enough and Claim 48.

For the proof of (6.78), we first prove that

$$\forall \rho > 0, \exists C_0(\rho) > 0, \forall \epsilon > 0, \|\hat{\Phi}_\epsilon\|_{C^0(\Gamma(\rho))} \leq C_0(\rho). \quad (6.81)$$

We now prove (6.81). Let $0 \leq i \leq n$. Up to change $\hat{\phi}_\epsilon^i$ into $-\hat{\phi}_\epsilon^i$, there exists a subsequence $\{x_\epsilon\}$ of points in $\Gamma(\rho)$ such that $\hat{\phi}_\epsilon^i(x_\epsilon) = \sup_{\Gamma(\rho)} |\hat{\phi}_\epsilon^i|$. We set $\delta_\epsilon = d_\epsilon^+(x_\epsilon, \text{supp}(\hat{v}_\epsilon))$ and we let $y_\epsilon \in \text{supp}(\hat{v}_\epsilon)$ be such that $\delta_\epsilon = |x_\epsilon - y_\epsilon|$. We divide the proof into 3 cases :

CASE 1 - $\delta_\epsilon^{-1} = O(1)$. Then, $\{e^{\hat{u}_\epsilon}\}$ is uniformly bounded in $I_{\min(\frac{\delta_\epsilon}{2}, \frac{\rho}{2})}(x_\epsilon)$ by estimates on the heat kernel (see (6.9)). By (6.77), $\hat{\phi}_\epsilon^i$ is bounded in $L^2(\Gamma(\frac{\rho}{2}))$. By elliptic theory for the Dirichlet-to-Neumann operator (see [109], Chapter 7.11, page 37), in $W^{1,2}(\Gamma(\frac{\rho}{2}))$, $\hat{\phi}_\epsilon^i$ is bounded (see (6.68)), and $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded by Sobolev embeddings.

CASE 2 - $\delta_\epsilon = O\left(\frac{\sqrt{\epsilon}}{\alpha_\epsilon}\right)$. Using Claim 41, we get that $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded.

CASE 3 - $\delta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\frac{\sqrt{\epsilon}}{\alpha_\epsilon \delta_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. We set

$$e^{w_\epsilon(x)} = \delta_\epsilon e^{2\hat{u}_\epsilon(x_\epsilon + \delta_\epsilon x)}$$

$$\psi_\epsilon(x) = \hat{\phi}_\epsilon^i(x_\epsilon + \delta_\epsilon x)$$

$$z_\epsilon = \frac{1}{\delta_\epsilon}(x_\epsilon - y_\epsilon)$$

so that

$$\begin{cases} \Delta \psi_\epsilon = 0 & \text{in } \mathbb{D}_5^+ \\ \partial_t \psi_\epsilon = -\sigma_\epsilon e^{w_\epsilon} \psi_\epsilon & \text{on } I_5. \end{cases} \quad (6.82)$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R} \times \{0\}$ with $|z_0| = 1$ such that $z_\epsilon \rightarrow z_0$ as $\epsilon \rightarrow 0$. By estimates (6.9), there is $D_1 > 0$ such that

$$e^{w_\epsilon} \leq D_1 \text{ in } I_{\frac{1}{2}}.$$

By Claim 41, since $y_\epsilon \in \text{supp}(\hat{v}_\epsilon)$, $\psi_\epsilon(z_\epsilon) = O(1)$ as $\epsilon \rightarrow 0$.

We first assume that ψ_ϵ does not vanish in $\mathbb{D}_3^+(0)$. Since $\psi_\epsilon(0) > 0$, $\psi_\epsilon > 0$ in $\mathbb{D}_3^+(0)$. Then, by Harnack's inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}^+, \psi_\epsilon(x) \geq D_2 \psi_\epsilon(0).$$

Since ψ_ϵ is positive, ψ_ϵ is weakly superharmonic in $\mathbb{D}_{|z_\epsilon|}^+(z_\epsilon) \subset \mathbb{D}_3^+(0)$ by (6.82) so that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{\pi |z_\epsilon|} \int_{\partial \mathbb{D}_{|z_\epsilon|}^+(z_\epsilon)} \psi_\epsilon d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}^+$, we get a constant $D_3 > 0$ such that $\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0)$. We conclude that $\hat{\phi}_\epsilon^i(x_\epsilon) = \psi_\epsilon(0)$ is bounded.

We now assume that ψ_ϵ vanishes in $\mathbb{D}_3^+(0)$. Since $X_r(\Omega(\rho), \xi, \Gamma(\rho), e^{i\epsilon}) = \emptyset$ by Claim 48, ψ_ϵ vanishes in $\mathbb{D}_4^+(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 17, Section 6.2.3, on $\Omega = \mathbb{D}_5^+(0)$ we get some constant $C_1 > 0$ such that

$$\int_{\mathbb{D}_4^+(0)} \psi_\epsilon^2 dx \leq C_1 \int_{\mathbb{D}_5^+(0)} |\nabla \psi_\epsilon|^2 dx.$$

By elliptic estimates on (6.82), $\{\psi_\epsilon\}$ is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}^+(0)$ and $\phi_\epsilon^i(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now prove (6.78). Let $\rho > 0$ and $0 \leq i \leq n$. Since $\hat{\phi}_\epsilon^i$ is harmonic in $\Omega(\frac{\rho}{2})$, by elliptic regularity theory, there exists a constant $K_0(\rho) > 0$ such that

$$\|\hat{\phi}_\epsilon^i\|_{C^0(\Omega(\rho))} \leq K_0(\rho) \left(\|\hat{\phi}_\epsilon^i\|_{L^2(\Omega(\frac{\rho}{2}))} + \|\hat{\phi}_\epsilon^i\|_{C^0(\Gamma(\frac{\rho}{2}))} \right)$$

and setting $C_2(\rho) = K_0(\rho) (C_1(\frac{\rho}{2}) + C_0(\frac{\rho}{2}))$ gives (6.78).

As in the proof of Claim 43, Claim 48 gives some capacity estimates and we get (6.79), and (6.80) is a consequence of (6.78), (6.79) and the equation (6.68). \diamond

We now need an estimate of $\{\Phi_\epsilon\}$ on the whole surface in order to prove later that no energy is lost in the necks.

Claim 50. For any $\rho > 0$, there exists a constant $C_0(\rho) > 0$ such that

$$\begin{aligned} \forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i, \rho) \cup \bigcup_{i=0}^t \bigcup_{j=1}^{s_i} \Omega_{i,j} \right), \\ |\Phi_\epsilon(x)| \leq C_0(\rho) \left(\ln \left(1 + \frac{d_g(x, \bar{a}_\epsilon)}{\alpha_\epsilon} \right) + 1 \right), \end{aligned}$$

where

$$\tilde{\Omega}_{i,j}^l = \omega_i^\epsilon \mathbb{D}_\rho(p_{i,j}) + a_\epsilon \text{ and } \bar{a}_\epsilon = \exp_{g_l, x_l}^{-1}(a_\epsilon).$$

Proof. Let $0 < \rho < \frac{1}{20R_0}$ and let $r > 0$ which satisfies the conclusion of Claim 48 for this ρ .

STEP 1 : We prove that for $0 \leq i \leq t$, there exists $A_i(\rho) > 0$ such that for all $0 \leq \beta \leq n$, for all sequence $\{\gamma_\epsilon\}$ with $\frac{\omega_i^\epsilon}{\rho} \leq \gamma_\epsilon \leq \rho \omega_{i+1}^\epsilon$, either

$$\forall x \in \mathbb{A}_{12R_0\rho}, \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right| \leq A_i(\rho)$$

or

$$\forall x, y \in \mathbb{A}_{12R_0\rho}, \frac{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|}{A_i(\rho)} \leq \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right| \leq A_i(\rho) \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|.$$

We let $\widetilde{A}_i(\rho)$ be equal to

$$\max_{0 \leq \beta \leq n} \sup_{\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho \omega_{i+1}^\epsilon} \sup_{\epsilon > 0} \left\{ \min \left\{ \max_{x \in J_{10R_0\rho}} \left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right|, \max_{x,y \in \mathbb{A}_{10R_0\rho}} \frac{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x) \right|}{\left| \overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y) \right|} \right\} \right\},$$

where we recall that for $r > 0$ $J_r = \mathbb{A}_r \cap \mathbb{R} \times \{0\}$. We assume by contradiction that $\widetilde{A}_i(\rho) = +\infty$. Then there exist $0 \leq \beta \leq n$, $\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho \omega_{i+1}^{\epsilon_m}$ such that $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$ and

$$\min \left\{ \max_{x \in J_{10R_0\rho}} \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|, \max_{x,y \in \mathbb{A}_{10R_0\rho}} \frac{\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|}{\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(y) \right|} \right\} \rightarrow +\infty \text{ as } m \rightarrow +\infty.$$

Let $x_m \in J_{10R_0\rho}$ be such that $\left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) \right| = \max_{x \in J_{10R_0\rho}} \left| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x) \right|$. We set

$$\delta_m = d(x_m, \text{supp}(\overline{v_{\epsilon_m}}^{\gamma_{\epsilon_m}}))$$

and take $y_m \in \text{supp}(\overline{v_{\epsilon_m}}^{\gamma_{\epsilon_m}})$ such that $|x_m - y_m| = \delta_m$. We study 3 cases each one leading to a contradiction.

CASE 1 - $\delta_m = O\left(\frac{\sqrt{\epsilon_m}}{\gamma_{\epsilon_m}}\right)$. We apply Claim 41 for the sequence of points $\{\exp_{g_l, x_l}(\gamma_{\epsilon_m} x_m + a_{\epsilon_m})\}_m$ in ∂M and we get a contradiction.

CASE 2 - $\delta_m \rightarrow 0$ and $\frac{\delta_m \gamma_{\epsilon_m}}{\sqrt{\epsilon_m}} \rightarrow +\infty$ as $m \rightarrow +\infty$. We set

$$e^{w_m} = \delta_m e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}}(x_m + \delta_m x),$$

$$\psi_m = \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m + \delta_m x) \text{ and}$$

$$z_m = \frac{1}{\delta_m}(y_m - x_m)$$

so that

$$\begin{cases} \Delta \psi_m = 0 \\ \partial_t \psi_m = -\sigma_{\epsilon_m} e^{w_m} \psi_m. \end{cases}$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R} \times \{0\}$ with $|z_0| = 1$ such that $z_m \rightarrow z_0$ as $m \rightarrow +\infty$. By (6.9), there is $D_1 > 0$ such that

$$e^{2w_m} \leq D_1 \text{ on } I_{\frac{1}{2}}.$$

By Claim 41, since $y_m \in \text{supp}(\overline{v_{\epsilon_m}}^{\gamma_{\epsilon_m}})$, $\psi_m(z_m) = O(1)$ as $m \rightarrow +\infty$.

We first assume that ψ_m does not vanish in $\mathbb{D}_3^+(0)$. Up to take $-\psi_m$, we can assume that $\psi_m > 0$ on $\mathbb{D}_3^+(0)$. Then, by Harnack inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}^+, \psi_m(x) \geq D_2 \psi_m(0).$$

Since ψ_m is positive, ψ_m is weakly superharmonic in $\mathbb{D}_{|z_m|}^+(z_m) \subset \mathbb{D}_3^+(0)$. Then,

$$\psi_m(z_m) \geq \frac{1}{\pi |z_m|} \int_{\partial \mathbb{D}_{|z_m|}(z_m)} \psi_m d\sigma$$

and keeping the part of the integral which lies in $\mathbb{D}_{\frac{1}{4}}^+$, we get a constant $D_3 > 0$ such that $\psi_m(z_m) \geq D_3 \psi_m(0)$. We conclude that $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

We assume now that ψ_m vanishes in $\mathbb{D}_3^+(0)$. By Claim 48, ψ_m vanishes in $\mathbb{D}_4^+(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 17 on $\Omega = \mathbb{D}_5^+(0)$, we get a Poincaré inequality

$$\int_{\mathbb{D}_4^+(0)} \psi_m^2 dx \leq C_1 \int_{\mathbb{D}_5^+(0)} |\nabla \psi_m|^2 dx .$$

By elliptic regularity theory, ψ_m is uniformly bounded on $\mathbb{D}_{\frac{1}{4}}^+(0)$ and $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

CASE 3 - $\frac{1}{\delta_m} = O(1)$. Up to the extraction of a subsequence, we assume that $x_m \rightarrow x$ in $J_{10R_0\rho}$ as $m \rightarrow +\infty$.

We first assume that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}$ vanishes in $\mathbb{A}_{5R_0\rho}$. We get by Claim 48 and the corollary of Theorem 17 on $\Omega = \mathbb{A}_{2R_0\rho}$, a constant $C_r > 0$ such that

$$\int_{\mathbb{A}_{4R_0\rho}} \psi_m^2 dx \leq C_r \int_{\mathbb{A}_{2R_0\rho}} |\nabla \psi_m|^2 dx .$$

By (6.9), there are some constants $\tilde{r} > 0$ and $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ on } I_{\tilde{r}}(x) .$$

By elliptic estimates, $\{\psi_m\}$ is uniformly bounded on $\mathbb{A}_{5R_0\rho} \cap \mathbb{D}_{\frac{\tilde{r}}{2}}(x)$ which gives a contradiction.

We assume now that $\psi_m := \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}$ does not vanish in $\mathbb{A}_{5R_0\rho}$. Up to take $-\psi_m$, we assume that $\psi_m > 0$ on $\mathbb{A}_{5R_0\rho}$.

Let's assume that $y_m \rightarrow y$ as $m \rightarrow +\infty$ with $y \in J_{7R_0\rho}$. By Claim 41, $\psi_m(y_m) = O(1)$. By (6.9), there exists a constant $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } I_{\delta-\tilde{\delta}}(x) ,$$

where $\tilde{\delta} = \min\left(\frac{\delta}{4}, \frac{R_0\rho}{4}\right)$. By Harnack's inequality, there exists $D_2 > 0$ such that

$$\forall z \in \mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}^+(x), \psi_m(x_m) \leq D_2 \psi_m(z) .$$

By weak superharmonicity on $\mathbb{D}_{3\tilde{\delta}}^+(y_m) \subset \mathbb{A}_{5R_0\rho}$,

$$\psi_m(y_m) \geq \frac{1}{\pi \times 3\tilde{\delta}} \int_{\partial \mathbb{D}_{3\tilde{\delta}}^+(y_m)} \psi_m d\sigma$$

We keep the part of the integral which lies in $\mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}^+$. Since the length of $\partial\mathbb{D}_{3\tilde{\delta}}^+(y_m) \cap \mathbb{A}_{6R_0\rho} \cap \mathbb{D}_{\delta-2\tilde{\delta}}^+$ is uniformly bounded from below, we get a constant $D_3 > 0$ such that $\psi_m(y_m) \geq D_3\psi_m(x_m)$. Then, $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(x_m) = O(1)$ which is absurd.

Assume now that $y_m \in \mathbb{R} \times \{0\} \setminus J_{8R_0\rho}$. By (6.9), there is a constant $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } \mathbb{A}_{9R_0\rho}.$$

By Harnack inequality, there exists a constant $C_1 > 0$ such that

$$\forall z, \tilde{z} \in \mathbb{A}_{10R_0\rho}, \frac{|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(\tilde{z})|}{C_1} \leq |\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m)| \leq C_1 |\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(z)|$$

which also leads to a contradiction.

We get $\widetilde{A}_i(\rho) < +\infty$. We now let $A_i(\rho)$ be equal to

$$\max_{0 \leq \beta \leq n} \sup_{\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho\omega_{i+1}^\epsilon} \sup_{\epsilon > 0} \left\{ \min \left\{ \max_{x \in \mathbb{A}_{12R_0\rho}} |\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)|, \max_{x,y \in \mathbb{A}_{12R_0\rho}} \frac{|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)|}{|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y)|} \right\} \right\}$$

and we assume by contradiction that $A_i(\rho) = +\infty$. Let γ_{ϵ_m} with $\frac{\omega_i^{\epsilon_m}}{\rho} \leq \gamma_{\epsilon_m} \leq \rho\omega_{i+1}^{\epsilon_m}$ and $\epsilon_m \rightarrow 0$ as $m \rightarrow +\infty$ be such that

$$\min \left\{ \max_{x \in \mathbb{A}_{12R_0\rho}} |\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)|, \max_{x,y \in \mathbb{A}_{12R_0\rho}} \frac{|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)|}{|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(y)|} \right\} \rightarrow +\infty \text{ as } m \rightarrow +\infty.$$

Then, by elliptic estimates there is some constant $K(\rho)$ such that

$$\max_{x \in \mathbb{A}_{12R_0\rho}} |\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)| \leq K(\rho) \left(\max_{x \in J_{10R_0\rho}} |\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)| + \left\| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}} \right\|_{L^2(\mathbb{A}_{10R_0\rho})} \right)$$

so that since $\widetilde{A}_i(\rho) < +\infty$,

$$\left\| \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}} \right\|_{L^2(\mathbb{A}_{10R_0\rho})} \rightarrow +\infty \text{ as } m \rightarrow +\infty.$$

By Poincaré inequalities given by the corollary of Theorem 17 on $\Omega = \mathbb{A}_{5R_0\rho}$, and by Claim 48, we clearly have that $\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}$ does not vanish in $\mathbb{A}_{5R_0\rho}$ and by Harnack inequalities,

$$\sup_{m \geq 0} \max_{x,y \in \mathbb{A}_{10R_0\rho}} \frac{|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)|}{|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(y)|} < +\infty$$

which contradicts the fact that $A_i(\rho) = +\infty$. Then $A_i(\rho) < +\infty$ and we get Step 1.

STEP 2 : We have that for $1 \leq i \leq t$, there exists $B_i(\rho) > 0$ such that for all $0 \leq \beta \leq n$, either

$$\forall x \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}), \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right| \leq B_i(\rho)$$

or

$$\forall x, y \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}), \frac{\left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(y) \right|}{B_i(\rho)} \leq \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(x) \right| \leq B_i(\rho) \left| \overline{\phi_\epsilon^\beta}^{\omega_i^\epsilon}(y) \right|$$

and there exists $B_{t+1}(\rho) > 0$ such that for all $0 \leq \beta \leq n$, either

$$\forall x \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho), \left| \phi_\epsilon^\beta(x) \right| \leq B_{t+1}(\rho)$$

or

$$\forall x, y \in M \setminus \bigcup_{i=1}^s B_g(p_i, \rho), \frac{\left| \phi_\epsilon^\beta(y) \right|}{B_{t+1}(\rho)} \leq \left| \phi_\epsilon^\beta(x) \right| \leq B_{t+1}(\rho) \left| \phi_\epsilon^\beta(y) \right|.$$

The proof of Step 2 follows exactly that of Step 1. Notice that if $m_0(\rho) > 0$, the third inequality holds by Claim 43. We leave the details to the reader.

STEP 3 : We prove that there exists $K_i(\rho) > 0$ such that for $0 \leq i \leq t$, and for all $x \in \mathbb{D}_{\tau_{i+1}^\epsilon}^+ \setminus \mathbb{D}_{t_i^\epsilon}^+$,

$$|F_\epsilon|(x) \leq K_i(\rho) \left\{ \max_{\partial \mathbb{D}_{t_i^\epsilon}^+} |F_\epsilon| + \ln \left(\frac{|x|}{t_i^\epsilon} \right) \right\} \quad (6.83)$$

where $t_i^\epsilon = 12R_0\omega_i^\epsilon$, $\tau_{i+1}^\epsilon = \frac{\omega_{i+1}^\epsilon}{12R_0}$ and $F_\epsilon(x) = \widetilde{\Phi}_\epsilon^l(a_\epsilon + x)$.

Let $0 \leq \beta \leq n$. We set

$$N_i^\epsilon = \{t_i^\epsilon \leq t \leq \tau_i^\epsilon; \exists x \in \mathbb{R}^2, |x| = t \text{ and } F_\epsilon(x) = 0\}.$$

Then, by the Courant Nodal theorem, N_i^ϵ has a finite number of connected components, bounded by $k+1$, since each connected component adds at least one nodal domain for the eigenfunction Φ_ϵ^β . By Step 1, we clearly have that

$$\forall x \in \mathbb{R}^2; |x| \in N_i^\epsilon \Rightarrow \left| F_\epsilon^\beta(x) \right| \leq A_i(\rho). \quad (6.84)$$

We let

$$c_{i,1}^\epsilon < d_{i,1}^\epsilon < c_{i,2}^\epsilon < d_{i,2}^\epsilon < \dots < c_{i,q_\epsilon}^\epsilon < d_{i,q_\epsilon}^\epsilon$$

be such that

$$N_i^\epsilon = [t_i^\epsilon, \tau_i^\epsilon] \setminus \bigcup_{j=1}^{q_\epsilon}]c_{i,j}^\epsilon, d_{i,j}^\epsilon[$$

with $\{q_\epsilon\}$ a bounded sequence of integers. Let $1 \leq j \leq q_\epsilon$. Then, F_ϵ^β does not vanish on $\mathbb{D}_{d_{i,j}^\epsilon}^+ \setminus \mathbb{D}_{c_{i,j}^\epsilon}^+$, and we can assume that $F_\epsilon^\beta > 0$ up to take $-F_\epsilon^\beta$. By the eigenvalue equation, F_ϵ^β is then weakly superharmonic on $\mathbb{D}_{d_{i,j}^\epsilon}^+ \setminus \mathbb{D}_{c_{i,j}^\epsilon}^+$. We set

$$f_\epsilon(u) = \frac{\int_{\partial \mathbb{D}_u^+} F_\epsilon^\beta(x) d\sigma(x)}{\pi u}.$$

Then,

$$\begin{aligned} f'_\epsilon(u) &= \frac{\int_{\partial \mathbb{D}_u} \partial_\nu F_\epsilon^\beta(x) d\sigma(x)}{\pi u} \\ &= \frac{-\int_{\mathbb{D}_u} \Delta F_\epsilon^\beta(x) dx + \int_{I_u} \partial_t F_\epsilon^\beta(s, 0) ds}{\pi u} \\ &= \frac{\int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s, 0) ds + \int_{I_u \setminus I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s, 0) ds}{\pi u} \end{aligned}$$

so that

$$f_\epsilon(u) = f_\epsilon(c_{i,j}^\epsilon) + \frac{\int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s, 0) ds}{\pi} \ln \left(\frac{u}{c_{i,j}^\epsilon} \right) + \int_{c_{i,j}^\epsilon}^u \frac{\int_{I_v \setminus I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s, 0) ds}{\pi v} dv.$$

By a Hölder inequality,

$$\left| \int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s, 0) ds \right| \leq \left(\int_{\partial M} (\phi_\epsilon^\beta)^2 e^{u_\epsilon} d\sigma_g \right)^{\frac{1}{2}} \left(\int_{\partial M} e^{u_\epsilon} d\sigma_g \right)^{\frac{1}{2}} \leq 1$$

and since F_ϵ^β is positive on $I_{d_{i,j}^\epsilon} \setminus I_{c_{i,j}^\epsilon}$,

$$f_\epsilon(u) \leq f_\epsilon(c_{i,j}^\epsilon) + \frac{1}{\pi} \ln \left(\frac{u}{c_{i,j}^\epsilon} \right) \text{ for } c_{i,j}^\epsilon \leq u \leq d_{i,j}^\epsilon.$$

By the second condition of Step 1, we have for $c_{i,j}^\epsilon \leq u \leq d_{i,j}^\epsilon$ that

$$\forall x \in \partial \mathbb{D}_u^+, F_\epsilon^\beta(x) \leq A_i(\rho) f_\epsilon(u).$$

Gathering these inequalities, for $1 \leq j \leq q_\epsilon$, we get a constant $K_i(\rho) > 0$ such that,

$$\forall x \in \partial \mathbb{D}_u^+, |F_\epsilon^\beta(x)| \leq K_i(\rho) \left(\max_{\partial \mathbb{D}_{t_i^\epsilon}^+} |F_\epsilon^\beta| + \ln \left(\frac{u}{t_i^\epsilon} \right) \right), \quad (6.85)$$

which is exactly Step 3.

We are now in position to prove the claim. By Step 2, we get some constant $L_i(\rho) > 0$ such that for $1 \leq i \leq t$,

$$\sup_{\mathbb{D}_{t_i^\epsilon}^+ \setminus \mathbb{D}_{\tau_i^\epsilon}^+} |F_\epsilon| \leq L_i(\rho) \left(\inf_{\mathbb{D}_{t_i^\epsilon}^+ \setminus \mathbb{D}_{\tau_i^\epsilon}^+} |F_\epsilon| + 1 \right) \quad (6.86)$$

and we get some constant $L_{t+1}(\rho)$ such that

$$\sup_{M(\rho)} |\Phi_\epsilon| \leq L_{t+1}(\rho) \left(\max_{\partial \mathbb{D}_{t+1}^+} |F_\epsilon| + 1 \right). \quad (6.87)$$

By (6.78) in Claim 49,

$$\sup_{\mathbb{D}_{t_0}^+} |F_\epsilon| \leq C_2 \left(\frac{1}{12R_0} \right). \quad (6.88)$$

Gathering (6.83), (6.86), (6.87) and (6.88), we get the claim. \diamond

In the following claim, we aim at passing to the limit in the equation (i) and the condition (ii) given by proposition 6 at the scale α_ϵ . The limiting function would then satisfy (6.92) and (6.93).

Claim 51. *We have that*

— *For any $\rho > 0$, there exists $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that*

$$\forall x \in \Gamma(\rho), |\hat{\Phi}_\epsilon|^2(x) \geq 1 - \beta_\epsilon. \quad (6.89)$$

— *For $\rho > 0$ and $x \in \Gamma(\rho)$, we set $\hat{\Psi}_\epsilon = \frac{\hat{\Phi}_\epsilon}{|\hat{\Phi}_\epsilon|}$. Then for any $\rho > 0$, $\{\hat{\Psi}_\epsilon\}$ is uniformly equicontinuous on $\mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$.*

— *For any $\rho > 0$, up to the extraction of a subsequence of $\{\hat{\Phi}_\epsilon\}$, there exist functions $\hat{\Phi} \in W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}) \cap L^\infty(\Gamma(\rho), \mathbb{R}^{n+1})$ and $\hat{\Psi} \in W^{\frac{1}{2},2}(\Gamma(\rho), \mathbb{S}^n) \cap \mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$ such that*

$$\hat{\Phi}_\epsilon \rightharpoonup \hat{\Phi} \text{ in } W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}) \quad (6.90)$$

and

$$\hat{\Psi}_\epsilon \rightarrow \hat{\Psi} \text{ in } \mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n) \text{ as } \epsilon \rightarrow 0 \quad (6.91)$$

with

$$|\hat{\Phi}|^2 \geq_{a.e.} 1 \text{ and } \hat{\Psi} = \frac{\hat{\Phi}}{|\hat{\Phi}|} \text{ on } \Gamma(\rho) \quad (6.92)$$

and for $0 \leq i \leq n$,

$$\begin{cases} \Delta \hat{\phi}^i = 0 & \text{in } \Omega(\rho) \\ \partial_t \hat{\phi}^i = -\sigma_k(M, [g]) \hat{\psi}^i d\hat{\nu} & \text{on } \Gamma(\rho) \end{cases} \quad (6.93)$$

in a weak sense.

Proof.

STEP 1 : We recall that $a_\epsilon \rightarrow a$ as $\epsilon \rightarrow 0$ with $\widetilde{z_{i_0}} = a$.

For $1 \leq j \leq s_0$ and $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)} \alpha_\epsilon^2}$,

$$\sup_{x \in \Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz = O(e^{-\frac{\rho^2}{8\theta_\epsilon}}). \quad (6.94)$$

For $0 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\bar{v}_l(a)}(\omega_i^\epsilon)^2}$,

$$\sup_{x \in \Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z) \right|^2 \overline{p}_\epsilon^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} x \right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}). \quad (6.95)$$

For $1 \leq i \leq s$ and $i \neq i_0$,

$$\sup_{x \in \Gamma(\rho)} \int_{I_\epsilon(p_i, \frac{\rho}{10})} |\Phi_\epsilon(z)|^2 p_\epsilon(\check{x}, z) d\sigma_g(z) = O(e^{-\frac{\rho^2}{8\epsilon}}). \quad (6.96)$$

Note that (6.96) was already proved in Step 1 of Claim 44. Note also that the proof of (6.94) reduces to (6.95) for $i = 0$. Let $0 \leq i \leq t$ and $1 \leq j \leq s_i$. Then, for $y \in \Gamma(\rho)$,

$$\begin{aligned} e^{\frac{\rho^2}{8\tau_i^\epsilon}} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z) \right|^2 \overline{p}_\epsilon^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} y \right) dz \\ \leq \frac{\int_{I_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z) \right|^2 e^{\overline{u}_\epsilon^{\omega_i^\epsilon}}}{\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u}_\epsilon^{\omega_i^\epsilon}}} \times O \left(\frac{e^{-\frac{\rho^2}{4\tau_i^\epsilon}(\frac{9}{10^2} - \frac{1}{2} - \frac{1}{100})}}{\sqrt{\tau_i^\epsilon}} \right) \\ \leq \frac{C_0}{\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u}_\epsilon^{\omega_i^\epsilon}}} \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}} \end{aligned}$$

where we used the uniform bound (6.8) on $\overline{p}_\epsilon^{\omega_i^\epsilon}$ on $\mathbb{D}_{\frac{1}{\rho}} \times \mathbb{D}_{\frac{1}{\rho}}$. We assume by contradiction that

$$\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u}_\epsilon^{\omega_i^\epsilon}} \leq \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}}.$$

Let $y \in \partial M$ be such that $\bar{y}^{\omega_i^\epsilon} \in I_{\frac{\rho}{10}}(p_{i,j})$. Then,

$$\begin{aligned} e^{\overline{u}_\epsilon^{\omega_i^\epsilon}(\bar{y}^{\omega_i^\epsilon})} &= e^{\bar{v}_l(y)} \omega_i^\epsilon \int_{\partial M} p_\epsilon(x, y) d\nu_\epsilon(y) \\ &\geq \int_{I_{\frac{\rho}{10}}(p_{i,j})} \overline{p}_\epsilon^{\omega_i^\epsilon}(z, \bar{y}^{\omega_i^\epsilon}) d\overline{v}_\epsilon^{\omega_i^\epsilon}(z) \\ &\geq \alpha_0 \frac{e^{-\frac{\rho^2}{80\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}} \int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{v}_\epsilon^{\omega_i^\epsilon} \end{aligned}$$

so that the assumption leads to

$$\int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{v}_\epsilon^{\omega_i^\epsilon} \leq \frac{e^{-\frac{\rho^2}{16\tau_i^\epsilon}}}{\alpha_0}.$$

For $z \in I_{\frac{\rho}{20}}(p_{i,j})$,

$$\begin{aligned} e^{\overline{u}_\epsilon^{\omega_i^\epsilon}(z)} &\leq \frac{\int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{v}_\epsilon^{\omega_i^\epsilon} + O\left(e^{-\frac{\rho^2}{4\tau_i^\epsilon}\left(\frac{1}{20^2} - \frac{1}{1000}\right)}\right)}{\sqrt{\tau_i^\epsilon}} \\ &\leq \frac{e^{-\frac{\rho^2}{16\tau_i^\epsilon}} + O\left(e^{-\frac{3\rho^2}{8000\tau_i^\epsilon}}\right)}{\alpha_0 \sqrt{\tau_i^\epsilon}}. \end{aligned}$$

Then, $e^{\overline{u}_\epsilon^{\omega_i^\epsilon}} \rightarrow 0$ uniformly on $\mathcal{C}^0(I_{\frac{\rho}{20}}(p_{i,j}))$ as $\epsilon \rightarrow 0$ and

$$\sigma_\star(\mathbb{D}_{\frac{\rho}{20}}(p_{i,j}), \zeta, I_{\frac{\rho}{20}}(p_{i,j}), e^{\overline{u}_\epsilon^{\omega_i^\epsilon}} \zeta) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

This contradicts (6.72) in Claim 48. The proof of Step 1 is now complete.

STEP 2 : There exists a sequence $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that

$$\forall x, y \in \Gamma(\rho), |x - y| \leq \frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon} \Rightarrow |\hat{\Phi}_\epsilon(x) - \hat{\Phi}_\epsilon(y)| \leq \beta_\epsilon. \quad (6.97)$$

We set $\gamma_\epsilon = \|\sqrt{\theta_\epsilon} e^{\hat{u}_\epsilon}\|_{\mathcal{C}^0(\Gamma(\rho))}^{\frac{1}{2}}$. We have that $\gamma_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. Indeed, for $r > 0$, and $x \in \Gamma(\rho)$,

$$\sqrt{\theta_\epsilon} e^{\hat{u}_\epsilon(x)} \leq \left(\frac{A_0}{\sqrt{4\pi}} + o(1) \right) \int_{I_r(x)} d\hat{v}_\epsilon + o(1) \leq \frac{A_0 \hat{v}(I_r(x))}{4\pi} + o(1) \leq \frac{A_0 D_1(\rho)}{\sqrt{4\pi} \ln(\frac{1}{r})} + o(1)$$

since we have (6.69) and thanks successively to (6.9), (6.8) and to (6.70), (6.79). We also have $\frac{\gamma_\epsilon}{\sqrt{\theta_\epsilon}} \rightarrow +\infty$ as $\epsilon \rightarrow 0$ since $\frac{\theta_\epsilon^{\frac{1}{4}}}{\gamma_\epsilon} = \|e^{\hat{u}_\epsilon}\|_{\mathcal{C}^0(\Gamma(\rho))}^{-\frac{1}{2}} \leq m_i(\rho)^{-\frac{1}{3}}$ is bounded and we have (6.69). Let x_ϵ and $y_\epsilon \in \Gamma(\rho)$ with $|x_\epsilon - y_\epsilon| \leq \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon}$. We set

$$F_\epsilon(z) = \hat{\Phi}_\epsilon(x_\epsilon + \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon} z)$$

and α_ϵ the mean value of F_ϵ in \mathbb{D}_3^+ . Then, we get constants $D_0, D, D' > 0$ such that

$$\begin{aligned} \|F_\epsilon - \alpha_\epsilon\|_{L^\infty(I_2(0))} &\leq D_0 \|F_\epsilon - \alpha_\epsilon\|_{H^1(I_2(0))} \\ &\leq D \|\partial_\nu F_\epsilon\|_{L^\infty(I_3(0))} + D \|F_\epsilon - \alpha_\epsilon\|_{L^2(\mathbb{D}_3^+(0))} \\ &\leq D \|\hat{\Phi}_\epsilon\|_{L^\infty(\Gamma(\rho))} \sigma_\epsilon \gamma_\epsilon + D' \|\nabla F_\epsilon\|_{L^2(\mathbb{D}_3^+(0))} \\ &\leq DC_2(\rho) \sigma_\epsilon \gamma_\epsilon + D' \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}} \end{aligned}$$

thanks successively to (6.80) and (6.78). See also Step 2 in the Proof of Claim 44. Setting

$$\beta_\epsilon = 2DC_2(\rho)\sigma_\epsilon\gamma_\epsilon + 2D' \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}},$$

$\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and we get Step 2.

STEP 3 : There exists a sequence $\beta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that for all $x \in \partial M$,

$$\hat{x} \in \Gamma(\rho) \Rightarrow \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon \quad (6.98)$$

and

$$\hat{x} \in \Gamma(\rho) \cap \text{supp}(\hat{v}_\epsilon) \Rightarrow |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon \quad (6.99)$$

Note that (6.98) gives (6.89) for $x \in \text{supp}(v_\epsilon)$ by Proposition 6. Let $x \in \partial M$ be such that $\hat{x} \in \Gamma(\rho)$.

$$\begin{aligned} \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| &\leq \int_{\partial M} p_\epsilon(x, y) \left| |\Phi_\epsilon(x)|^2 - |\Phi_\epsilon(y)|^2 \right| d\sigma_g(y) \\ &\leq \int_{I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) \left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - |\hat{\Phi}_\epsilon(z)|^2 \right| dz \\ &\quad + I_\epsilon \\ &\quad + \sum_{i \neq i_0} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2 p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + \sum_{i=0}^t \sum_{j=1}^{s_i} \int_{I_\rho(p_{i,j})} \left| \overline{\Phi_\epsilon}^{\omega_i^\epsilon} \right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} \hat{x} \right) dz, \end{aligned}$$

where

$$I_\epsilon = \int_{\partial M \setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} p_\epsilon(x, y) \left(C_2^2(\rho) + C_0^2(\rho) \left(\ln \left(1 + \frac{d_g(y, \bar{a}_\epsilon)}{\alpha_\epsilon} \right) + 1 \right)^2 \right) d\sigma_g(y)$$

Here, we used Claim 49 and Claim 50. By (6.94), (6.95), (6.96) and (6.97),

$$\left| |\hat{\Phi}_\epsilon(\hat{x})|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq 2C_2(\rho)\beta_\epsilon + O(e^{-\frac{\rho^2}{8\alpha_\epsilon^2}}) + I_\epsilon$$

and there are some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that

$$\begin{aligned} I_\epsilon &\leq K_0(\rho) \ln \left(\frac{\delta(\partial M)}{\alpha_\epsilon} \right)^2 \int_{\partial M \setminus \Gamma_l} p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + K_1(\rho) \int_{\hat{\Gamma}_l \setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) (\ln(1 + |z|)^2 + 1) dz. \end{aligned}$$

Since $\frac{\alpha_\epsilon^2}{\epsilon} \rightarrow +\infty$ as $\epsilon \rightarrow 0$,

$$\begin{aligned} \ln \left(\frac{\delta(\partial M)}{\alpha_\epsilon} \right)^2 \int_{\partial M \setminus \Gamma_l} p_\epsilon(x, y) d\sigma_g(y) &\leq \ln \left(\frac{\delta(\partial M)}{\alpha_\epsilon} \right)^2 \times O \left(\frac{e^{-\frac{\delta(M)^2}{4\epsilon}}}{\sqrt{\epsilon}} \right) \\ &= o(1) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

and by (6.6),

$$\begin{aligned} \int_{\hat{\Gamma}_l \setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z, \hat{x}) (\ln(1 + |z|)^2 + 1) dz \\ \leq \int_{\mathbb{R} \setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}} \frac{A_0}{\sqrt{4\pi\theta_\epsilon}} e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}} (\ln(1 + |z|)^2 + 1) dz \\ \leq \int_{\mathbb{R} \setminus I_{\frac{1}{\beta_\epsilon}}(0)} \frac{A_0}{\sqrt{4\pi}} e^{-\frac{|y|^2}{8}} \left(\ln \left(1 + \left| \hat{x} + \sqrt{\theta_\epsilon} y \right| \right)^2 + 1 \right) dy \\ = o(1) \text{ uniformly for } \hat{x} \in \Gamma(\rho). \end{aligned}$$

Up to increase β_ϵ , we get (6.98). The same estimates can be obtained for $|\Phi_\epsilon|$ instead of $|\Phi_\epsilon|^2$, and we get up to increase β_ϵ for $x \in \partial M$ such that $\hat{x} \in \Gamma(\rho)$,

$$||\hat{\Phi}_\epsilon(\hat{x})| - K_\epsilon[|\Phi_\epsilon|](x)| \leq \beta_\epsilon.$$

Since, if $z \in \text{supp}(\hat{v}_\epsilon) \cap \Gamma(\rho)$, we have

$$||\hat{\Phi}_\epsilon(z)|^2 - 1| \leq \beta_\epsilon,$$

up to increase β_ϵ , we get for $x \in \partial M$ such that $\hat{x} \in \text{supp}(\hat{v}_\epsilon) \cap \Gamma(\rho)$,

$$|K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon.$$

We follow Step 4 in the proof of Claim 44 to prove that $\hat{\Psi}_\epsilon$ is uniformly equicontinuous on $\Gamma(\rho)$. Indeed, we can use the corollary of Theorem 17 thanks to Claim 48. Therefore, up to the extraction of a subsequence, $\hat{\Psi}_\epsilon \rightarrow \hat{\Psi}$ in $\mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$ as $\epsilon \rightarrow 0$.

STEP 4 : We have that

$$\hat{\phi}_\epsilon^i e^{\hat{u}_\epsilon} ds \rightarrow_\star \hat{\psi}^i \hat{v} \text{ in } \mathcal{M}(\Gamma(\rho)) \text{ as } \epsilon \rightarrow 0.$$

Let $\zeta \in \mathcal{C}_c^0(I(\rho))$ and $R > \frac{1}{\rho}$. Then

$$\begin{aligned}
 & \int_{\mathbb{R} \times \{0\}} \zeta(z) \left(\hat{\phi}_\epsilon^i(z) e^{\hat{u}_\epsilon(z)} dz - \hat{\psi}^i(z) d\hat{v}(z) \right) \\
 &= \int_{\partial M \setminus I_R} \left(\int_{\Gamma(\rho)} p_\epsilon(x, y) \zeta(y) \phi_\epsilon^i(y) d\sigma_g(y) \right) dv_\epsilon(x) \\
 &+ \int_{I_R} \left(\int_{I_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 &+ \int_{\Gamma(\rho)} \zeta(x) \left(\int_{I_R} (\hat{\psi}_\epsilon^i(z) - \hat{\psi}^i(x)) |\hat{\Phi}_\epsilon(z)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 &+ \int_{\Gamma(\rho)} \zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{I_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 &+ \int_{\Gamma(\rho)} \left(\zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{I_R} \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) - \zeta(x) \hat{\psi}^i(x) d\hat{v}(x) \right).
 \end{aligned}$$

We have by (6.10) that

$$\begin{aligned}
 & \int_{\partial M \setminus I_R} \left(\int_{\Gamma(\rho)} p_\epsilon(x, y) \zeta(y) \phi_\epsilon^i(y) d\sigma_g(y) \right) dv_\epsilon(x) \\
 & \leq \|\zeta\|_\infty C_2(\rho) \sup_{y \in \partial M \setminus I_R} \int_{I_{\frac{1}{\rho}}} p_\epsilon(x, y) d\sigma_g(x) \\
 & = o(1) \text{ as } \epsilon \rightarrow 0.
 \end{aligned}$$

By Step 1, Claim 50 and (6.8),

$$\begin{aligned}
 & \int_{I_R} \left(\int_{I_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 & \leq \sup_{x \in I_R} \int_{I_R} |\zeta(z) - \zeta(x)| |\hat{\phi}_\epsilon^i(z)| \hat{p}_\epsilon(z, x) dz \\
 & \leq \sum_{j=1}^{s_0} \sup_{x \in I_R} |\zeta(x)| \int_{I_{\frac{\rho}{10}}(p_{0,j})} |\hat{\phi}_\epsilon^i(z)| \hat{p}_\epsilon(z, x) dz \\
 & + \sup_{x \in I_R} \int_{I_R \setminus \bigcup_{j=1}^{s_0} I_{\frac{\rho}{10}}(p_{0,j})} |\zeta(z) - \zeta(x)| |\hat{\phi}_\epsilon^i(z)| \hat{p}_\epsilon(z, x) dz \\
 & \leq \|\zeta\|_\infty \sum_{j=1}^{s_0} \sup_{x \in \Gamma(\rho)} \left(\int_{I_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz \right)^{\frac{1}{2}} \\
 & + C_0(\rho) (1 + \ln(1 + C_0 R)) \sup_{x \in I_R} \int_{\mathbb{R} \times \{0\}} |\zeta(z) - \zeta(x)| \frac{e^{\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}} dz \\
 & = o(1) \text{ as } \epsilon \rightarrow 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\Gamma(\rho)} \zeta(x) \left(\int_{I_R} \left(\hat{\psi}_\epsilon^i(z) - \hat{\psi}_\epsilon^i(x) \right) |\hat{\Phi}_\epsilon(z)| \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 & \leq 2 \|\zeta\|_\infty \sup_{x \in \Gamma(\rho)} \sum_{j=1}^{s_0} \left(\int_{I_{\frac{\rho}{10}}(p_{0,j})} |\hat{\Phi}_\epsilon(z)|^2 \hat{p}_\epsilon(z, x) dz \right)^{\frac{1}{2}} \\
 & \quad + \|\zeta\|_\infty C_2 \left(\frac{\rho}{10} \right) \sup_{x \in \Gamma(\rho)} \int_{\Gamma(\frac{\rho}{10})} \left| \hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z) \right| \hat{p}_\epsilon(z, x) dz \\
 & \quad + 2 \|\zeta\|_\infty C_0(\rho) (1 + \ln(1 + C_0 R)) \sup_{x \in \Gamma(\rho)} \int_{I_R \setminus \Gamma(\frac{\rho}{10})} \hat{p}_\epsilon(z, x) dz \\
 & = o(1) \text{ as } \epsilon \rightarrow 0,
 \end{aligned}$$

where by (6.8),

$$\begin{aligned}
 & \sup_{x \in \Gamma(\rho)} \int_{\Gamma(\frac{\rho}{10})} \left| \hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z) \right| \hat{p}_\epsilon(z, x) dz \\
 & \leq \sup_{x \in \Gamma(\rho)} \int_{\Gamma(\frac{\rho}{10})} \left| \hat{\psi}_\epsilon^i(x) - \hat{\psi}_\epsilon^i(z) \right| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}} dz \\
 & = o(1) \text{ as } \epsilon \rightarrow 0.
 \end{aligned}$$

We also have that

$$\begin{aligned}
 & \int_{\Gamma(\rho)} \zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{I_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \right) d\hat{v}_\epsilon(x) \\
 & \leq \|\zeta\|_\infty \sup_{x \in \Gamma(\rho) \cap \text{supp}(\hat{v}_\epsilon)} \int_{I_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz.
 \end{aligned}$$

We use (6.99) of Step 3, in order to prove that

$$\sup_{x \in \Gamma(\rho) \cap \text{supp}(\hat{v}_\epsilon)} \int_{I_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, x) dz \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (6.100)$$

Let $x \in \partial M$ be such that $\hat{x} \in \Gamma(\rho) \cap \text{supp}(\hat{v}_\epsilon)$,

$$\begin{aligned}
 K_\epsilon[|\Phi_\epsilon|](x) - 1 &= \int_{\partial M \setminus I_R} (|\Phi_\epsilon(y)| - 1) p_\epsilon(x, y) d\sigma_g(y) \\
 & \quad + \int_{I_R} (|\hat{\Phi}_\epsilon(z)| - 1) \hat{p}_\epsilon(z, \hat{x}) dz
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \int_{\partial M \setminus \check{I}_R} (|\Phi_\epsilon(y)| - 1) p_\epsilon(x, y) d\sigma_g(y) \right| \\
 & \leq \int_{\partial M \setminus \Gamma_I} p_\epsilon(x, y) d\sigma_g(y) K_0(\rho) \ln \left(\frac{\delta(\partial M)}{\alpha_\epsilon} \right) \\
 & \quad + K_1(\rho) \int_{\hat{\Gamma}_I \setminus I_R} \hat{p}_\epsilon(z, \hat{x}) (1 + \ln(1 + |z|)) dz \\
 & \leq O \left(\frac{e^{-\frac{\delta(\partial M)^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} \ln \left(\frac{\delta(\partial M)}{\alpha_\epsilon} \right) \right) \\
 & \quad + K_1(\rho) \int_{\mathbb{R} \setminus I_R} A_0 \frac{e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}} (1 + \ln(1 + |z|)) dz \\
 & \leq O \left(\int_{\mathbb{R} \setminus I_{\frac{R}{\sqrt{\theta_\epsilon}}}} e^{-\frac{|y|^2}{8}} (1 + \ln(1 + |\hat{x} + \sqrt{\theta_\epsilon} y|)) dz \right) \\
 & = o(1) \text{ as } \epsilon \rightarrow 0.
 \end{aligned}$$

This gives (6.100). By (6.11),

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \left| \int_{I_R} \hat{p}_\epsilon(z, x) dz - 1 \right| = 0,$$

so that

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\Gamma(\rho)} \left(\zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{I_R} \hat{p}_\epsilon(z, x) dz \right) d\hat{\nu}_\epsilon(x) - \zeta(x) \hat{\psi}^i(x) d\hat{\nu}(x) \right) \right) = 0.$$

Gathering all these computations, we get Step 4.

As a conclusion, (6.98) in Step 3 gives (6.89) for $x \in \text{supp}(\nu_\epsilon)$ by Proposition 6. In the remark before Step 4, we get (6.91). Then, (6.89), (6.90) and (6.91) give (6.92). We finally get (6.93) passing to the limit in the equation satisfied by $\hat{\phi}_\epsilon^i$ thanks to Step 4. This ends the proof of the Claim.

◇

Thanks to Claim 51, a diagonal extraction gives some functions $\hat{\Phi} : \mathbb{R}_+^2 \setminus \{p_{0,1}, \dots, p_{0,s_0}\} \rightarrow \mathbb{R}^{n+1}$ and $\hat{\Psi} : \mathbb{R} \setminus \{p_{0,1}, \dots, p_{0,s_0}\} \rightarrow \mathbb{S}^n$ such that for any $\rho > 0$, the conclusions (6.90), (6.91), (6.92) and (6.93) of Claim 51 hold true for $\hat{\Phi}$ and $\hat{\Psi}$.

We now give energy estimates on these limit functions which will be useful at the end of the proof. We recall that $\lambda : \mathbb{D} \setminus \{p\} \rightarrow \mathbb{R}_+^2$ is defined page 209. We set $\check{\Phi} = \hat{\Phi} \circ \lambda : \mathbb{D} \setminus \{p, q_0, \dots, q_{s_0}\}$ and $\check{\Psi} = \hat{\Psi} \circ \lambda : \mathbb{S}^1 \setminus \{p, q_0, \dots, q_{s_0}\}$, where $q_j = \lambda^{-1}(p_{0,j}) \in \mathbb{S}^1$ and we set

$$D(\rho) = \mathbb{D} \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{i=1}^{s_0} \mathbb{D}_\rho(q_i) \right) \text{ and } S(\rho) = \mathbb{S}^1 \cap D(\rho).$$

We Let $\check{\nu}$ be the measure without atom on \mathbb{S}^1 such that

$$e^{\hat{\mu}_\epsilon} d\theta \rightharpoonup_\star d\check{\nu} \text{ in } \mathcal{M}(S(\rho)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$. It is equal to $\lambda^*(\hat{v})$ outside $\{p, q_0, \dots, q_{s_0}\}$.

We also set some function ω on \mathbb{D} which satisfies the following equation

$$\begin{cases} \Delta\omega = 0 & \text{in } \mathbb{D} \\ \omega = |\check{\Phi}| & \text{on } S^1 \end{cases} \quad (6.101)$$

in a weak sense. Such a harmonic function exists since $|\check{\Phi}| \in W^{\frac{1}{2},2}(S^1)$ and we have $\omega \in W^{1,2}(\mathbb{D})$.

Claim 52.

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{D(\rho)} |\nabla \check{\Phi}_\epsilon|^2 dx \geq \int_{\mathbb{D}} \frac{|\nabla \check{\Phi}|^2}{\omega} dx \geq \sigma_k(M, [g]) \int_{S^1} d\check{v} + \int_{\mathbb{D}} \frac{|\check{\Phi}|^2 |\nabla \omega|^2}{\omega^3} dx \quad (6.102)$$

where $\int_{S^1} d\check{v} \geq m_i$.

Proof. Let $\eta \in C_c^\infty(D(\sqrt{\rho}))$ be given by Claim 40 with $\eta \geq 1$ on $D(\rho)$ and

$$\int_{\mathbb{D}} |\nabla \eta|^2 \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

By the weak maximum principle on (6.101),

$$\inf_{\mathbb{D}} \omega \geq \inf_{S^1} |\check{\Phi}| \geq 1$$

and by the same computations as in the proof of Claim 45,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{D(\rho)} |\nabla \check{\Phi}_\epsilon|^2 dx &\geq \int_{D(\rho)} |\nabla \check{\Phi}|^2 dx \\ &\geq \int_{\mathbb{D}} \eta \frac{|\nabla \check{\Phi}|^2}{\omega} dx \\ &\geq \sigma_k(M, [g]) \int_{S^1} \eta d\check{v} + \int_{\mathbb{D}} \eta \frac{|\check{\Phi}|^2}{\omega^3} |\nabla \omega|^2 \\ &\quad - \sum_{i=0}^n \int_{\mathbb{D}} \frac{\check{\Phi}_i}{\omega} \langle \nabla \eta, \nabla \check{\Phi}_i \rangle - \int_{\mathbb{D}} \langle \nabla \eta, \nabla \omega \rangle \frac{|\check{\Phi}|^2}{2\omega^2} + \frac{1}{2} \int_{\mathbb{D}} \langle \nabla \eta, \nabla \omega \rangle \\ &\geq \sigma_k(M, [g]) \int_{S^1} \eta d\check{v} + \int_{\mathbb{D}} \eta \frac{|\check{\Phi}|^2}{\omega^3} |\nabla \omega|^2 - \frac{C'}{\sqrt{\ln\left(\frac{1}{\rho}\right)}} \end{aligned}$$

where C' is a constant independent of ρ . Indeed, $\check{\Phi}_i, \omega \in W^{1,2}(\mathbb{D})$ and we have for $0 \leq i \leq n$ that

$$\Delta(\omega - \check{\Phi}_i) = 0 \text{ and } \Delta(\omega + \check{\Phi}_i) = 0$$

in a weak sense. By the weak maximum principle (see [43], Theorem 8.1),

$$\inf_{\mathbb{D}} (\omega - \check{\Phi}_i) \geq \inf_{S^1} (\omega - \check{\Phi}_i) \geq 0$$

and

$$\inf_{\mathbb{D}} (\omega + \check{\phi}_i) \geq \inf_{\mathbb{S}^1} (\omega + \check{\Phi}) \geq 0$$

since $|\check{\phi}_i| \leq |\check{\Phi}| \leq \omega$ on \mathbb{S}^1 . Then,

$$\sup_{\mathbb{D}} \frac{|\check{\phi}_i|}{\omega} \leq 1 \text{ and } \sup_{\mathbb{D}} \frac{|\check{\Phi}|^2}{\omega^2} \leq n + 1 .$$

We finally get (6.102), passing to the limit as $\rho \rightarrow 0$. We have that $\int_{\mathbb{S}^1} d\check{\nu} \geq m_i$ thanks to (6.59), (6.61) and (6.76). This ends the proof of the claim.

◇

6.6.2 Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$

We now assume that $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$, we let $\theta_0 = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{e^{2\check{\nu}_l(a)} \alpha_\epsilon}$ and we denote by $\hat{\nu}$ the weak* limit of $\hat{\nu}_\epsilon$ in $\mathcal{M}(\mathbb{R} \times \{0\})$. Let $R_0 > 0$ and $x \in I_{R_0}$. We have by (6.8) that

$$\begin{aligned} e^{\hat{u}_\epsilon}(x) &= e^{v_l(\check{x})} \alpha_\epsilon \int_{\partial M} p_\epsilon(\check{x}, y) d\nu_\epsilon(y) \\ &\leq \frac{A_0 e^{v_l(\check{x})} \alpha_\epsilon}{\sqrt{4\pi\epsilon}} \int_{\partial M} d\nu_\epsilon \\ &\leq \frac{A_0}{\sqrt{4\pi\epsilon}} (1 + o(1)) . \end{aligned}$$

Since $m_i > 0$, we get that $\theta_0 < +\infty$. Now, we let \hat{u} be a smooth function on $\mathbb{R} \times \{0\}$ defined by

$$e^{\hat{u}(x)} = \int_{\mathbb{R} \times \{0\}} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu}(y) . \quad (6.103)$$

Let $R_0 > 0$, $R > R_0$ and $x \in I_{R_0}$. We have

$$\begin{aligned}
 \left| e^{\hat{u}_\epsilon(x)} - e^{\hat{u}(x)} \right| &= \left| \int_{\partial M} \alpha_\epsilon p_\epsilon(\check{x}, y) d\nu_\epsilon(y) - e^{\hat{u}(x)} \right| \\
 &\leq \int_{\partial M \setminus I_R} \alpha_\epsilon p_\epsilon(\check{x}, y) d\nu_\epsilon(y) \\
 &\quad + \left| \int_{I_R} \hat{p}_\epsilon(x, y) d\hat{\nu}_\epsilon(y) - \int_{\mathbb{R} \times \{0\}} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu}(y) \right| \\
 &= o(1) + \frac{A_0}{\sqrt{4\pi\theta_0}} (1 + o(1)) e^{-\frac{(R-R_0)^2}{8\theta_0}} \\
 &\quad + \left| \int_{I_R} \left(\hat{p}_\epsilon(x, y) - \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} \right) d\hat{\nu}_\epsilon \right| \\
 &\quad + \left| \int_{I_R} \frac{e^{-\frac{|x-y|^2}{8\theta_0}}}{\sqrt{4\pi\theta_0}} (d\hat{\nu}_\epsilon - d\hat{\nu}) \right| + \int_{\mathbb{R} \times \{0\} \setminus I_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu} \\
 &\rightarrow \frac{A_0}{\sqrt{4\pi\theta_0}} e^{-\frac{(R-R_0)^2}{8\theta_0}} + \int_{\mathbb{R} \times \{0\} \setminus I_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu} \text{ as } \epsilon \rightarrow 0.
 \end{aligned}$$

Letting $R \rightarrow +\infty$, we get for any $R_0 > 0$ that

$$e^{\hat{u}_\epsilon} \rightarrow e^{\hat{u}} \text{ in } \mathcal{C}^0(I_{R_0}) \text{ as } \epsilon \rightarrow 0. \quad (6.104)$$

With Claim 41, $\{\hat{\phi}_\epsilon^i\}$ is bounded in $L^2(I_R)$ for any $R > 0$. With (6.104) and elliptic estimates on the Dirichlet-to-Neumann operator (see [109], Chapter 7.11, page 37)

$$\begin{cases} \Delta \hat{\phi}_\epsilon^i = 0 & \text{in } \mathbb{D}_{R_0}^+ \\ \partial_t \hat{\phi}_\epsilon^i = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\phi}_\epsilon^i & \text{on } I_{R_0}, \end{cases}$$

we get some smooth function $\hat{\Phi}$ on \mathbb{R}_+^2 such that for any $R_0 > 0$,

$$\hat{\phi}_\epsilon^i \rightarrow \hat{\phi}^i \text{ in } \mathcal{C}^1(\mathbb{D}_{R_0}^+) \text{ as } \epsilon \rightarrow 0. \quad (6.105)$$

and

$$\begin{cases} \Delta \hat{\phi}^i = 0 & \text{in } \mathbb{R}_+^2 \\ \partial_t \hat{\phi}^i = -\sigma_k(M, [g]) e^{\hat{u}} \hat{\phi}^i & \text{on } \mathbb{R} \times \{0\}. \end{cases} \quad (6.106)$$

We now prove the following

Claim 53. *We have the following energy inequality*

$$\int_{\mathbb{R}_+^2} |\nabla \hat{\Phi}(x)|^2 dx \geq \sigma_k(M, [g]) \int_{S^1} e^{\hat{u}} d\theta, \quad (6.107)$$

where $e^{\hat{u}} = e^{\hat{u}} \circ \lambda$

Proof.

STEP 1 : Up to the extraction of a subsequence, there exists some sequences $\{\omega_i^\epsilon\}$ with $0 \leq i \leq t+1$ and $0 \leq t \leq k$ and

$$\alpha_\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \dots \ll \omega_{t+1}^\epsilon = \delta_0$$

and for $1 \leq i \leq t$ and $1 \leq j \leq s_i$ some points $p_{i,j} \in J_{\frac{1}{R_0}}$ with $R_0 > 0$ and $s-1 + \sum_{i=1}^t s_i \leq k$ such that for all $\rho > 0$, there exists $C_0(\rho)$ such that

$$\forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i, \rho) \cup \bigcup_{i=1}^t \bigcup_{j=1}^{s_i} \Omega_{i,j} \right),$$

$$|\Phi_\epsilon|(x) \leq C_0(\rho) \left(\ln \left(1 + \frac{d_g(\bar{a}_\epsilon, x)}{\sqrt{\epsilon}} \right) + 1 \right)$$

where $\widetilde{\Omega}_{i,j} = \omega_\epsilon^i \mathbb{D}_\rho^+(p_{i,j}) + a_\epsilon$ and $\bar{a}_\epsilon = \exp_{g_l, x_l}^{-1}(a_\epsilon)$. We also have that for all $\rho > 0$,

$$\sup_{x \in \Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left| \overline{\Phi_\epsilon^{\omega_i^\epsilon}}(z) \right|^2 \overline{p_\epsilon^{\omega_i^\epsilon}} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} x \right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}). \quad (6.108)$$

for $1 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\bar{v}_l(a)}(\omega_i^\epsilon)^2}$ and

$$\sup_{x \in \Gamma(\rho)} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(z)|^2 p_\epsilon(\check{x}, z) dz = O(e^{-\frac{\rho^2}{8\epsilon}}). \quad (6.109)$$

For $1 \leq i \leq s$ and $i \neq i_0$.

For the estimate of Φ_ϵ , we follow the proof of Claim 48 and Claim 50, using (6.104) and (6.105) instead of the estimates of Claim 49. The proof of (6.108) and (6.109) follows the proof of Step 1 in Claim 51, which is a consequence of Claim 48.

STEP 2 : We have that

$$\int_{\mathbb{R} \times \{0\}} |\hat{\Phi}(y)|^2 \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dy \geq 1. \quad (6.110)$$

In order to prove (6.110), it suffices to use Proposition 6 and prove that for $R_0 > 0$ fixed, $x \in \partial M$ such that $\hat{x} \in I_{R_0}$, we have

$$\int_{\mathbb{R} \times \{0\}} |\hat{\Phi}(y)|^2 \frac{e^{-\frac{|\hat{x}-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} - K_\epsilon[|\Phi_\epsilon|^2](x) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (6.111)$$

Let's prove (6.111). We fix $r > 0$ and $R > r$. Let $x \in M$ be such that $\hat{x} \in I_r$. We fix $\rho > 0$. Then,

$$\begin{aligned} \left| K_\epsilon[|\Phi_\epsilon|^2](x) - \int_{I_R} |\hat{\Phi}(z)|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dz \right| &= \int_{\partial M \setminus \check{I}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + \int_{I_R} \hat{p}_\epsilon(z, \hat{x}) |\hat{\Phi}_\epsilon(z)|^2 dz \\ &\quad - \int_{I_R} |\hat{\Phi}(z)|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dz. \end{aligned}$$

There exist some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that, by Step 1,

$$\begin{aligned} \int_{\partial M \setminus \check{I}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) d\sigma_g(y) &\leq K_0(\rho) \int_{\partial M \setminus \Gamma_l} \ln \left(\frac{\delta(\partial M)}{\sqrt{\epsilon}} \right)^2 p_\epsilon(x, y) d\sigma_g(y) \\ &\quad + K_1(\rho) \int_{\hat{\Gamma}_l \setminus I_R} (\ln(1 + |z|)^2 + 1) \hat{p}_\epsilon(z, \hat{x}) dz \\ &\quad + \sum_{i=1}^t \sum_{j=1}^{s_i} \int_{I_{\frac{\rho}{10}}(p_{i,j})} |\overline{\Phi}_\epsilon^{\omega_i^\epsilon}(z)|^2 \overline{p}_\epsilon^{\omega_i^\epsilon} \left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} \hat{x} \right) dz \\ &\quad + \sum_{i \neq i_0} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x, y) d\sigma_g(y) \\ &\leq O \left(\ln \left(\frac{\delta(\partial M)}{\sqrt{\epsilon}} \right)^2 \frac{e^{-\frac{\delta(\partial M)^2}{4\epsilon}}}{\sqrt{\epsilon}} \right) \\ &\quad + O \left(e^{-\frac{\rho^2}{8\tau_1^\epsilon}} \right) \\ &\quad + \frac{K_1(\rho) A_0}{\sqrt{\pi\theta_0}} \int_{\mathbb{R} \times \{0\} \setminus I_R} (\ln(1 + |z|)^2 + 1) e^{-\frac{|\hat{x}-z|^2}{8\theta_0}} dz. \end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$ and then as $R \rightarrow +\infty$, we get (6.111) and then (6.110). This ends the proof of Step 2.

STEP 3 : We have that

$$\sigma_k(M, [g]) \int_{\mathbb{R} \times \{0\}} |\hat{\Phi}(y)|^2 e^{\hat{u}(y)} dy \leq \int_{\mathbb{R}_+^2} |\nabla \hat{\Phi}(x)|^2 dx. \quad (6.112)$$

By contradiction, we assume that there is $\epsilon_0 > 0$ such that

$$\sigma_k(M, [g]) \int_{\mathbb{R} \times \{0\}} |\hat{\Phi}(y)|^2 e^{\hat{u}(y)} dy \geq \int_{\mathbb{R}_+^2} |\nabla \hat{\Phi}(x)|^2 dx + \epsilon_0.$$

We fix $R > 0$. By the equation (6.106),

$$\begin{cases} \frac{1}{2} \Delta |\hat{\Phi}|^2 = -|\nabla \hat{\Phi}|^2 & \text{in } \mathbb{R}_+^2 \\ \frac{1}{2} \partial_t |\hat{\Phi}|^2 = -\sigma_k(M, [g]) e^{\hat{u}} |\hat{\Phi}|^2 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We integrate on \mathbb{D}_R^+ ,

$$-\frac{1}{2} \int_{\partial \mathbb{D}_R^+} \partial_\nu (|\hat{\Phi}|^2) d\sigma = \sigma_k(M, [g]) \int_{I_R} e^{\hat{u}} |\hat{\Phi}|^2 - \int_{\mathbb{D}_R^+} |\nabla \hat{\Phi}|^2 \geq \frac{\epsilon_0}{2}$$

for any $R > R_0$, for some $R_0 > 0$, since $e^{\hat{u}} |\hat{\Phi}|^2 \in L^1(\mathbb{R} \times \{0\})$ and $|\nabla \hat{\Phi}|^2 \in L^1(\mathbb{R}_+^2)$. We set

$$f(r) = \frac{\int_{\partial \mathbb{D}_r^+} |\hat{\Phi}|^2 d\sigma}{\pi r}.$$

Then, for $R > R_0$, $\pi f'(R) \leq -\frac{\epsilon_0}{R}$ so that

$$f(R) \leq -\frac{\epsilon_0}{\pi} \ln \left(\frac{R}{R_0} \right) + f(R_0) \rightarrow -\infty \text{ as } R \rightarrow +\infty$$

which contradicts the fact that $f(R) > 0$. This ends the proof of Step 3.

We are now in position to get the claim. We integrate (6.110) against \hat{v} and (6.103) against dx , and we obtain

$$\int_{\mathbb{R} \times \{0\}} |\hat{\Phi}(y)|^2 e^{\hat{u}(y)} dy \geq \int_{\mathbb{R} \times \{0\}} d\hat{v} = \int_{\mathbb{R} \times \{0\}} e^{\hat{u}(y)} dy \quad (6.113)$$

and we get (6.107) with (6.113) and (6.112). ◇

6.7 Proof of Theorem 15

6.7.1 Regularity of the limiting measures

In this subsection, we aim at proving the following no neck energy and regularity result, keeping the notations of Proposition 7.

Proposition 8. *For $i \in \{1, \dots, N\}$, there exists $q_{i,1}, \dots, q_{i,s_i} \in \mathbb{S}^1$ and $e^{\hat{u}_i} \in L^\infty(\mathbb{S}^1)$, smooth except maybe at one point, positive such that for all $\rho > 0$,*

$$e^{\hat{u}_i^\epsilon} d\theta \rightharpoonup_* e^{\hat{u}_i} d\theta \text{ on } \mathcal{M}(S_i(\rho)) \text{ as } \epsilon \rightarrow 0$$

with $S_i(\rho) = \mathbb{S}^1 \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{i=j}^{s_i} \mathbb{D}_\rho(q_{i,j}) \right)$ and $\int_{\mathbb{S}^1} e^{\hat{u}_i} d\theta = m_i$.

If $m_0 > 0$, there exists p_1, \dots, p_s and a density e^{u_0} on ∂M , smooth, such that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup_* e^{u_0} d\sigma_g \text{ on } \mathcal{M}(I(\rho)) \text{ as } \epsilon \rightarrow 0$$

with $M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$ and $\int_{\partial M} e^{u_0} d\sigma_g = m_0$.

Proof. Let \tilde{N} be such that for $1 \leq i \leq N$,

$$1 \leq i \leq \tilde{N} \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

and

$$\tilde{N} + 1 \leq i \leq N \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \text{ is bounded.}$$

We now reintroduce the indices i we dropped in section 6.6 :

For $1 \leq i \leq \tilde{N}$ fixed, we recall (see just before Claim 52) that we set

$$\{q_{i,1}, \dots, q_{i,s_i}\} = \{\lambda^{-1}(p_{0,1}), \dots, \lambda^{-1}(p_{0,s_0})\}$$

defined by Claim 48 and we recall that (6.76), that is $q_{i,1}, \dots, q_{i,s_i} \in \mathbb{R} \times \{0\}$ satisfy

$$Z\left(\mathbb{S}^1, \{e^{\tilde{u}_\epsilon^i} d\theta\}\right) \subset \{p, q_{i,1}, \dots, q_{i,s_i}\},$$

and that the notations before Claim 52 hold :

$$D_i(\rho) = \mathbb{D} \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(q_{i,j}) \right) \text{ and } S_i(\rho) = \mathbb{S}^1 \cap D_i(\rho)$$

and $\check{\nu}_i$ is the measure without atoms defined by

$$e^{\tilde{u}_\epsilon^i} d\theta \rightharpoonup_\star \check{\nu}_i \text{ in } \mathcal{M}(S_i(\rho)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$.

For $\tilde{N} + 1 \leq i \leq N$, the notations just before Claim 53 define $e^{\tilde{u}_i}$ and $e^{\check{u}_i}$ as

$$e^{\hat{u}_\epsilon^i} \rightarrow e^{\hat{u}_i} \text{ in } \mathcal{C}^0(I_{\frac{1}{\rho}}) \text{ as } \epsilon \rightarrow 0 \text{ and}$$

$$e^{\check{u}_\epsilon^i} \rightarrow e^{\check{u}_i} \text{ in } \mathcal{C}^0(\mathbb{S}^1 \setminus \mathbb{D}_\rho(p)) \text{ as } \epsilon \rightarrow 0$$

for any $\rho > 0$. Notice that $e^{\check{u}_i} = e^{\hat{u}_i} \circ \lambda$.

We also have $\{p_1, \dots, p_s\}$ such that (6.55) holds and denote

$$M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$$

and

$$I(\rho) = \partial M \setminus \bigcup_{i=1}^s I_g(p_i, \rho)$$

and ν_0 the measure without atoms such that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup_\star \nu_0 \text{ in } \mathcal{M}(I(\rho)) \text{ as } \epsilon \rightarrow 0.$$

Then, we have by (6.59) and (6.61) that

$$\int_{\mathbb{S}^1} d\check{\nu}_i \geq m_i \tag{6.114}$$

for $1 \leq i \leq \tilde{N}$ and

$$\int_{\mathbb{S}^1} e^{\check{u}_i} d\theta \geq m_i \tag{6.115}$$

and by (6.60) and (6.62) that

$$\int_{\partial M} d\nu_0 \geq m_0 . \quad (6.116)$$

Considering for $1 \leq i \leq N$ the set $M_i^\epsilon(\rho)$ such that

$$\left(H_{a_i^\epsilon, \alpha_i^\epsilon}\right)^{-1} \left(\widetilde{M_i^\epsilon(\rho)}^{l_i}\right) = \Omega_i(\rho) ,$$

(6.56), (6.61) give that

$$M(\rho) \cap M_i^\epsilon(\rho) = \emptyset \quad (6.117)$$

and (6.58) or (6.57) and (6.62) give that

$$i \neq j \Rightarrow M_i^\epsilon(\rho) \cap M_j^\epsilon(\rho) = \emptyset \quad (6.118)$$

for ϵ small enough.

By (6.117) and (6.118), we have for $\rho > 0$ and ϵ small enough

$$\int_M |\nabla \Phi_\epsilon|_g^2 dv_g \geq \mathbf{1}_{m_0 > 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g + \sum_{i=1}^N \int_{\Omega_i(\rho)} |\nabla \hat{\Phi}_\epsilon^i|^2 dx , \quad (6.119)$$

Then, applying (6.54) in Claim 45 if $m_0 > 0$, (6.102) in Claim 52 for $1 \leq i \leq \tilde{N}$, (6.105) and (6.107) in Claim 53 for $\tilde{N} + 1 \leq i \leq N$, (6.114), (6.116) and the conservation of the mass (6.63),

$$\sum_{i=0}^N m_i = 1 ,$$

we get from (6.119) that

$$\begin{aligned} \sigma_k(M, [g]) &= \lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_M |\nabla \Phi_\epsilon|_g^2 \\ &\geq \mathbf{1}_{m_0 > 0} \int_M \frac{|\nabla \Phi|_g^2}{\omega} dv_g + \sum_{i=1}^{\tilde{N}} \int_{\mathbb{D}} \frac{|\nabla \check{\Phi}_i|^2}{\omega_i} dx + \sum_{i=\tilde{N}+1}^N \int_{\mathbb{R}_+^2} |\nabla \hat{\Phi}_i|^2 dx \\ &\geq \mathbf{1}_{m_0 > 0} \left(\sigma_k(M, [g]) \int_{\partial M} d\nu_0 + \int_M \frac{|\Phi|^2 |\nabla \omega|_g^2}{\omega^3} dv_g \right) \\ &\quad + \sum_{i=1}^{\tilde{N}} \left(\sigma_k(M, [g]) \int_{\mathbb{S}^1} d\check{\nu}_i + \int_{\mathbb{D}} \frac{|\check{\Phi}_i|^2 |\nabla \omega_i|^2}{\omega_i^3} dx \right) \\ &\quad + \sum_{i=\tilde{N}+1}^N \sigma_k(M, [g]) \int_{\mathbb{S}^1} e^{\check{\nu}_i} d\theta \\ &\geq \sigma_k(M, [g]) + \mathbf{1}_{m_0 > 0} \int_M \frac{|\Phi|^2 |\nabla \omega|_g^2}{\omega^3} dv_g + \sum_{i=1}^{\tilde{N}} \int_{\mathbb{D}} \frac{|\check{\Phi}_i|^2 |\nabla \omega_i|^2}{\omega_i^3} dx . \end{aligned}$$

Therefore, all the inequalities are equalities in Claim 45, (6.116), Claim 52, (6.114) and Claim 53. Then, we get for $1 \leq i \leq \tilde{N}$ that $\omega_i = 1$ on \mathbb{D} so that

$$|\hat{\Phi}_i|^2 = 1 \text{ on } \mathbb{S}^1 ,$$

for $1 \leq i \leq \tilde{N}$ that

$$\int_{S^1} d\check{v}_i = m_i ,$$

and if $m_0 > 0$ that $\omega = 1$ so that

$$|\Phi|^2 = 1 \text{ on } \partial M$$

and

$$\int_{\partial M} d\nu_0 = m_0 .$$

Let $1 \leq i \leq \tilde{N}$. Then, $\hat{\Psi}_i = \hat{\Phi}_i$ on $\mathbb{R} \times \{0\}$ and the equation (6.93) gives that

$$\begin{cases} \Delta \hat{\Phi}_i = 0 \\ (-\partial_t) \hat{\Phi}_i = \sigma_k(M, [g]) \hat{\Phi}_i d\nu_i \end{cases}$$

in a weak sense on $\mathbb{R} \times \{0\} \setminus \{q_{i,1}, \dots, q_{i,s_i}\}$. Then, $d\hat{\nu}_i = \frac{\hat{\Phi}_i \cdot (-\partial_t) \hat{\Phi}_i}{\sigma_k(M, [g])} ds$ which means that $\hat{\nu}_i$ is absolutely continuous with respect to ds and

$$\begin{cases} |\hat{\Phi}_i|^2 = 1 & \text{in } \mathbb{R} \times \{0\} \\ (-\partial_t) \hat{\Phi}_i \wedge \hat{\Phi}_i = 0 & \text{in } \mathbb{R} \times \{0\} . \end{cases}$$

This means that $\hat{\Phi}_i$ is weakly $\frac{1}{2}$ -harmonic on $\mathbb{R}_+^2 \setminus \{q_{i,1}, \dots, q_{i,s_i}\}$. Then, by Da Lio (see [24], Proposition 2.2), since $\int_{\mathbb{R}_+^2} |\nabla \hat{\Phi}_i|^2 dx < +\infty$, we can extend $\hat{\Phi}_i$ as a $\frac{1}{2}$ -harmonic map on \mathbb{R}_+^2 . By the regularity theory for weakly $\frac{1}{2}$ -harmonic maps of Da Lio- Rivière, see [25], $\hat{\Phi}_i$ is smooth and $\frac{1}{2}$ -harmonic on \mathbb{R}_+^2 . Setting $e^{\hat{\nu}_i} = \frac{\hat{\Phi}_i \cdot (-\partial_t) \hat{\Phi}_i}{\sigma_k(M, [g])}$, and coming back to the disc, we get the first part of the claim for $1 \leq i \leq \tilde{N}$.

For $\tilde{N} + 1 \leq i \leq N$, the convergence (6.104) ends the proof of the first part of the proposition.

If $m_0 > 0$, then, $\Psi = \Phi$ and the equation (6.42) gives that

$$\begin{cases} \Delta_g \Phi = 0 \\ \partial_\nu \Phi = \sigma_k(M, [g]) \Phi d\nu \end{cases}$$

in a weak sense on $M \setminus \{p_1, \dots, p_s\}$. Then, $d\nu = \frac{\Phi \cdot \partial_\nu \Phi}{\sigma_k(M, [g])} d\sigma_g$ which means that ν is absolutely continuous with respect to $d\sigma_g$ and

$$\begin{cases} |\Phi|^2 = 1 & \text{in } \partial M \\ \partial_\nu \Phi \wedge \Phi = 0 & \text{in } \partial M . \end{cases}$$

This means that Φ is weakly harmonic on $M \setminus \{p_1, \dots, p_s\}$ with free boundary. Then, by Laurain-Petrides (see [70], Claim 4), since $\int_M |\nabla \Phi|^2 dv_g < +\infty$, we can extend Φ as a harmonic map on M with free boundary and Φ is smooth on M . The smoothness of weakly harmonic maps with free boundary was proved in [102] and [70]. Setting $e^\nu = \frac{\Phi \cdot \partial_\nu \Phi}{\sigma_k(M, [g])}$, we get the second part of the proposition.

◇

6.7.2 Gaps and no concentration

We prove now by contradiction that $N = 0$, so that the maximizing sequence $\{e^{\mu_\epsilon} d\sigma_g\}$ does not have any concentration points. Therefore, by Proposition 8 with $m_0 = 1$, the proof of Theorem 15 will follow.

We now assume that $N \geq 1$ and we use Proposition 8 and the gap assumption that (6.3) is strict in order to get a contradiction.

For $1 \leq i \leq N$, let θ_i be the maximal integer such that

$$\frac{\sigma_{\theta_i}(\mathbb{D})}{m_i} < \sigma_k(M, [g]) \quad (6.120)$$

and let θ_0 be the maximal integer such that

$$\frac{\sigma_{\theta_0}(M, [g])}{m_0} < \sigma_k(M, [g]) \quad (6.121)$$

if $m_0 > 0$. We set $\theta_0 = -1$ if $m_0 = 0$. We get that for $i \in \{1, \dots, N\}$,

$$\sigma_{\theta_i+1}(\mathbb{D}) \geq m_i \sigma_k(M, [g]) \quad (6.122)$$

and

$$\sigma_{\theta_0+1}(M, [g]) \geq m_0 \sigma_k(M, [g]) \quad (6.123)$$

Then, by the spectral gap assumption that (6.3) is strict, we have that

$$\sum_{i=0}^N (\theta_i + 1) \geq k + 1 \quad (6.124)$$

Indeed, if $\sum_{i=0}^N (\theta_i + 1) \leq k$, the spectral gap gives that

$$\sum_{i=1}^N \sigma_{\theta_i+1}(\mathbb{D}) + \sigma_{\theta_0+1}(M, [g]) < \sigma_k(M, [g])$$

and this contradicts (6.63) (6.122) and (6.123).

Now, we define at least $k + 1$ test functions for the min-max characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{\mu_\epsilon})$.

Let $1 \leq i \leq N$. We denote by $(\varphi_i^0, \dots, \varphi_i^{\theta_i})$ an orthonormal family in $L^2(\partial M, e^{\mu_0} dv_g)$ if $i = 0$ and in $L^2(S^1, e^{\mu_i} d\theta)$ if $i \neq 0$, such that if $0 \leq j \leq \theta_i$, φ_i^j is an eigenfunction for $\sigma_j(M, g, \partial M, e^{\mu_0})$ if $i = 0$ and for $\sigma_j(\mathbb{D}, \xi, S^1, e^{\mu_i})$ if $i \neq 0$. Such functions exist by Proposition 8 and lie in \mathcal{C}^1 .

We fix $\rho > 0$. We denote by η_i some function defined with Claim 40 by

- $\eta_0 \in \mathcal{C}_c^\infty(M(\sqrt{\rho}))$, $\eta_0 \geq 1$ on $M(\rho)$ and $\int_M |\nabla \eta_0|_g^2 dv_g \leq \frac{C}{\ln(\frac{1}{\rho})}$.
- If $i \neq 0$, $\eta_i \in \mathcal{C}_c^\infty(S_i(\sqrt{\rho}))$, $\eta_i \geq 1$ on $S_i(\rho)$ and $\int_{\mathbb{D}} |\nabla \eta_i|^2 dx \leq \frac{C}{\ln(\frac{1}{\rho})}$.

We set for $0 \leq i \leq N$ and $0 \leq j \leq \theta_i$ some test functions $\check{\zeta}_i^j$, defined by

$$\check{\zeta}_0^j = \eta_0 \varphi_0^j \text{ on } M$$

and if $i \neq 0$, $\check{\zeta}_i^j$ depends on ϵ and satisfies for any $\epsilon > 0$

$$\left(\check{\zeta}_i^j\right)_\epsilon^i = \eta_i \varphi_i^j \text{ on } \mathbb{D}$$

extended by 0 on M .

Note that all the test functions ξ_i^j lie in C^1 and are uniformly bounded. Note also that by (6.117) and (6.118), if ϵ small enough,

$$i \neq i' \Rightarrow \text{supp}(\xi_i^j) \cap \text{supp}(\xi_{i'}^{j'}) = \emptyset$$

for $i, i' \in \{0, \dots, N\}$, $0 \leq j \leq \theta_i$ and $0 \leq j' \leq \theta_{i'}$. For $1 \leq i \leq N$, we let E_i be the vectorspace spanned by $(\xi_i^0, \xi_i^1, \dots, \xi_i^{\theta_i})$ and with (6.124), we deduce by (6.4) that

$$\sigma_\epsilon \leq \max_{0 \leq i \leq N} \sup_{\xi \in E_i \setminus \{0\}} \frac{\int_M |\nabla \xi|_g^2 dv_g}{\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g}. \quad (6.125)$$

Let $i \in \{1, \dots, N\}$. For $\xi = \sum_{j=0}^{\theta_i} \mu_j \xi_i^j \in E_i$, with $\mu_j \in \mathbb{R}$ and $\sum_j \mu_j^2 = 1$, we get

$$\int_M |\nabla \xi|_g^2 dv_g = \int_{\mathbb{D}} \left| \nabla \left(\eta_i \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j \right) \right|^2 dx$$

and denoting $\varphi = \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j$, we have

$$\begin{aligned} \int_M |\nabla \xi|_g^2 dv_g &= \int_{\mathbb{D}} (\eta_i)^2 |\nabla \varphi|^2 dx + 2 \int_{\mathbb{D}} \eta_i \varphi \langle \nabla \eta_i, \nabla \varphi \rangle dx + \int_{\mathbb{D}} \varphi^2 |\nabla \eta_i|^2 dx \\ &\leq \int_{\mathbb{D}} |\nabla \varphi|^2 dx + 2 \|\eta_i \varphi\|_\infty \left(\int_{\mathbb{D}} |\nabla \varphi|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |\nabla \eta_i|^2 dx \right)^{\frac{1}{2}} \\ &\quad + \|\varphi\|_\infty^2 \int_{\mathbb{D}} |\nabla \eta_i|^2 dx \\ &\leq \int_{\mathbb{D}} |\nabla \varphi|^2 dx + O \left(\frac{1}{\sqrt{\ln(\frac{1}{\rho})}} \right) \text{ as } \rho \rightarrow 0. \end{aligned}$$

We also have that

$$\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g = \int_{S^1} \eta_i^2 \varphi^2 e^{u_\epsilon} d\theta.$$

By Proposition 8, we get that

$$\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g = \int_{S^1} \eta_i^2 \varphi^2 e^{u_\epsilon} d\theta + o(1) \text{ as } \epsilon \rightarrow 0$$

so that

$$\lim_{\epsilon \rightarrow 0} \int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g \geq \int_{S^1} \varphi^2 e^{u_\epsilon} d\theta + o(1) \text{ as } \rho \rightarrow 0.$$

The same work can be done for $\xi \in E_0$, so that passing to the limit as $\epsilon \rightarrow 0$ and then as $\rho \rightarrow 0$ in (6.125), we get

$$\sigma_k(M, [g]) \leq \max \left\{ \max_{1 \leq i \leq N} \sup_{\varphi \in E_i \setminus \{0\}} \frac{\int_{\mathbb{D}} |\nabla \varphi|^2 dx}{\int_{S^1} \varphi^2 e^{u_\epsilon} d\theta}, \sup_{\varphi \in E_0 \setminus \{0\}} \frac{\int_M |\nabla \varphi|_g^2 dv_g}{\int_{\partial M} \varphi^2 e^{u_\epsilon} d\sigma_g} \right\}$$

where F_i is the space spanned by $\varphi_i^0, \dots, \varphi_i^{\theta_i}$. Therefore,

$$\begin{aligned} \sigma_k(M, [g]) &\leq \max \left\{ \max_{1 \leq i \leq N} \sigma_{\theta_i}(\mathbb{D}, \xi, S^1, e^{\tilde{u}_i}), \sigma_{\theta_0}(M, g, \partial M, e^{u_0}) \right\} \\ &\leq \max \left\{ \max_{1 \leq i \leq N} \frac{\sigma_{\theta_i}(\mathbb{D})}{m_i}, \frac{\sigma_{\theta_0}(M, [g])}{m_0} \right\} \end{aligned}$$

which contradicts (6.122) and (6.123). Therefore, there is no concentration of $\{e^{u_\epsilon} d\sigma_g\}$.

Therefore, $N = 0$ and by Proposition 8 with $m_0 = 1$, Theorem 15 follows.

6.8 Proof of Theorem 14

We prove Theorem 14 in this section. Notice that light modifications of the proof allow us to prove that if (6.3) is strict, the set of maximal metrics for $\sigma_k(M, [g])$ is compact, and if we have that (6.2) is strict, the set of maximal metrics for $\sigma_k(\gamma, m)$ is compact.

Let $\gamma \geq 0$ and $m \geq 1$ be such that $(\gamma, m) \neq (0, 1)$ and $[g_\alpha]$ be a sequence of conformal classes on a compact oriented manifold of genus γ with m boundary components such that

$$\sigma_\alpha = \sigma_k(M, [g_\alpha]) \rightarrow \sigma_k(\gamma, m) \text{ as } \alpha \rightarrow +\infty, \quad (6.126)$$

where g_α denotes the unique metric in its conformal class such that

- The curvature of g_α is constant, equal to 0 if $(\gamma, m) = (0, 2)$, and -1 if $(\gamma, m) \neq (0, 2)$.
- The boundary ∂M of M is a union of closed geodesics with respect to g_α .

By the gap assumption that (6.2) is strict, we have in particular that

$$\sigma_k(M, [g_\alpha]) > \max_{\substack{1 \leq j \leq k \\ i_1 + \dots + i_s = j}} \sigma_{k-j}(M, [g_\alpha]) + \sum_{m=1}^s \sigma_{i_m}(\mathbb{D}^2, [\xi])$$

for α large enough. By Theorem 15, this gives some smooth harmonic maps with free boundary $\phi_\alpha : (M, g_\alpha) \rightarrow S^{n_\alpha}$ for some $n_\alpha > 0$, such that if \tilde{g}_α is a metric conformal to g_α with the induced metric on the boundary ∂M satisfying

$$d\sigma_{\tilde{g}_\alpha} = e^{u_\alpha} d\sigma_{g_\alpha},$$

where

$$e^{u_\alpha} = \frac{\Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha}{\sigma_\alpha},$$

then $\int_{\partial M} d\sigma_{\tilde{g}_\alpha} = 1$ and $\sigma_k(M, \tilde{g}_\alpha) = \sigma_k(M, [g_\alpha])$. Since the multiplicity of σ_k is bounded by a constant which only depends on k, γ and m (see [39] and [62]), we can assume that $n = n_\alpha$ is fixed.

We have the following quantification result on sequences of harmonic maps with free boundary by Laurain-Petrides, [70], Theorem 1 :

Proposition 9. *Let (M, g) be a smooth Riemannian surface with a smooth non empty boundary. We refer to the notations introduced in Section 6.2.1 for the metric g . Let $q_1, \dots, q_t \in M$. Let $\Phi_\alpha : (M_\alpha, g_\alpha) \rightarrow \mathbb{B}^{n+1}$ be an harmonic map with free boundary on an open set $M_\alpha \subset M$ such that*

- *For any $\rho > 0$, there exists $\alpha_\rho > 0$ such that for any $\alpha > \alpha_\rho$, $M_\alpha \supset M \setminus \bigcup_{i=1}^t B_g(q_i, \rho)$.*

- For any $\rho > 0$, $g_\alpha \rightarrow g$ in $M \setminus \bigcup_{i=1}^t B_g(q_i, \rho)$ as $\alpha \rightarrow +\infty$.
- $\Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha > 0$ on $M_\alpha \cap \partial M$ and

$$\limsup_{\alpha \rightarrow +\infty} \int_{M_\alpha \cap \partial M} \Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha} < +\infty$$

Then, up to the extraction of a subsequence, there exist

- Some harmonic map with free boundary $\Phi : M \rightarrow \mathbb{S}^n$.
- Sequences of points $p_\alpha^1, \dots, p_\alpha^s$ of ∂M converging to some points p^1, \dots, p^s of ∂M as $\alpha \rightarrow +\infty$ and sequences of scales $\delta_\alpha^1, \dots, \delta_\alpha^s$ converging to 0 as $\alpha \rightarrow +\infty$ such that

$$\frac{d_g(p_\alpha^i, p_\alpha^j)}{\delta_\alpha^i + \delta_\alpha^j} + \frac{\delta_\alpha^i}{\delta_\alpha^j} + \frac{\delta_\alpha^j}{\delta_\alpha^i} \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty \quad (6.127)$$

- Some harmonic extensions of non constant $\frac{1}{2}$ -harmonic maps, $\omega_1, \dots, \omega_s : \mathbb{D} \rightarrow \mathbb{B}^{n+1}$ such that

$$\int_M |\nabla \Phi|_g^2 dv_g + \sum_{i=1}^s \int_{\mathbb{D}} |\nabla \omega_i|^2 dx = \mathcal{E} \quad (6.128)$$

where

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \lim_{\alpha \rightarrow +\infty} \int_{\partial M \setminus \bigcup_{i=1}^t I_g(q_i, \rho)} \Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha}$$

and for all $\rho > 0$,

$$\Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha} \rightharpoonup_\star \Phi \cdot \partial_\nu \Phi d\sigma_g \text{ on } I(\rho), \quad (6.129)$$

$$\hat{\Phi}_\alpha^i \cdot (-\partial_t \hat{\Phi}_\alpha^i) ds \rightharpoonup_\star \hat{\omega}_i \cdot (-\partial_t \hat{\omega}_i) ds \text{ on } \Gamma_i(\rho), \quad (6.130)$$

where we define the sets

$$I(\rho) = \partial M \setminus \left(\bigcup_{i=1}^t I_g(q_i, \rho) \cup \bigcup_{z \in Z(\partial M \setminus \bigcup_{i=1}^t I_g(q_i, \rho), \Phi_\alpha \cdot \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha})} I_g(z, \rho) \right) \text{ and}$$

$$\Gamma_i(\rho) = I_{\frac{1}{\rho}} \setminus \bigcup_{z \in Z(I_{\frac{1}{\rho}}, \hat{\Phi}_\alpha^i \cdot (-\partial_t \hat{\Phi}_\alpha^i) ds)} I_\rho(z)$$

and the functions on \mathbb{R}_+^2

$$\hat{\Phi}_\alpha^i(x) = \widetilde{\Phi}_\alpha^{l_i}(\delta_\alpha^i x + \tilde{p}_\alpha^{l_i}) \text{ and } \hat{\omega}_i = \omega_i \circ \lambda^{-1},$$

where $1 \leq l_i \leq L$ is chosen such that $p^i \in \omega_{l_i}$ and λ is defined page 209.

Assuming that $g_\alpha \rightarrow g$ as $\alpha \rightarrow +\infty$ for some metric g with constant curvature and which defines closed geodesics boundary components, we apply Proposition 9 for $M_\alpha = M$, Φ_α , g_α and g . Notice that the use of Proposition 9 together with the gap assumption that (6.2) is strict follows exactly the same path as the use of Proposition 8 together with the gap assumption that (6.3) is strict in order to prove that the maximizing sequences do not have any concentration points. Therefore, one can easily contradict the fact that (6.2) is assumed to be strict in this case.

We assume now that the sequence of conformal classes $[g_\alpha]$ degenerates in the following sense :

- If $(\gamma, m) = (0, 2)$, in the case of the annulus, this means that $R_\alpha \rightarrow +\infty$ or $R_\alpha \rightarrow 1$, where $R_\alpha > 1$ denotes the real parameter such that (M, g_α) is isometric to $\mathbb{D}_{R_\alpha} \setminus \mathbb{D}$.
- If $(\gamma, m) \neq (0, 2)$, in the hyperbolic case, this means that the injectivity radius $i_{g_\alpha}(M) \rightarrow 0$ as $\alpha \rightarrow +\infty$ so that there exist closed geodesics which length goes to 0 or geodesics which cross two boundary components of (M, g_α) with length going to 0.

Let's tackle both cases in order to contradict that the gap (6.2) is strict. During all the proof, we identify \mathbb{R}^2 and \mathbb{C} thanks to the map $F(x, y) = x + iy$.

6.8.1 The case of the annulus

Let $(\gamma, m) = (0, 2)$. Then, (M, g_α) is isometric to $(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}, \xi)$.

We first assume that $R_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. We denote by $\Gamma_1 = \mathbb{S}^1$ and $\Gamma_2 = \mathbb{S}_{R_\alpha}^1$ the boundary components,

$$m_1 = \lim_{\alpha \rightarrow +\infty} \int_{\Gamma_1} e^{u_\alpha} d\sigma_\xi \text{ and } m_2 = \lim_{\alpha \rightarrow +\infty} \int_{\Gamma_2} e^{u_\alpha} d\sigma_\xi.$$

With the inversion $\iota(z) = \frac{1}{\bar{z}}$, we have $\iota(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}) = \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$, $\iota(\Gamma_1) = \mathbb{S}^1$ and the harmonic map with free boundary

$$\Phi_\alpha^1 = \Phi_\alpha \circ \iota : \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}} \rightarrow \mathbb{B}^{n+1}$$

satisfies the hypotheses of Proposition 9 on (\mathbb{D}, ξ) since $\mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$ exhausts \mathbb{D} . We have some limits $\Phi^1, \omega_1^1, \dots, \omega_{s_1}^1$ such that

$$\int_{\mathbb{D}} |\nabla \Phi^1|^2 dx + \sum_{i=1}^{s_1} \int_{\mathbb{D}} |\nabla \omega_i^1|^2 dx = m_1$$

and the conclusion of Proposition 9 holds for some associated scales.

With the dilatation $H(z) = \frac{z}{R_\alpha}$, we have $H(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}) = \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$, $H(\Gamma_2) = \mathbb{S}^1$ and the harmonic map with free boundary

$$\Phi_\alpha^2 = \Phi_\alpha \circ H^{-1} : \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}} \rightarrow \mathbb{B}^{n+1}$$

satisfies the hypotheses of Proposition 9 on (\mathbb{D}, ξ) since $\mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$ exhausts \mathbb{D} . We have some limits $\Phi^2, \omega_1^2, \dots, \omega_{s_2}^2$ such that

$$\int_{\mathbb{D}} |\nabla \Phi^2|^2 dx + \sum_{i=1}^{s_2} \int_{\mathbb{D}} |\nabla \omega_i^2|^2 dx = m_2$$

and the conclusion of Proposition 9 holds for some associated scales.

Following the proof of section 6.7.2, we use suitable eigenfunctions associated to the previous smooth limiting maps at their respective concentration scales as test functions for σ_α . They give a contradiction for the assumption that (6.2) is strict which reads as

$$\sigma_k(0, 2) > \max_{i_1 + \dots + i_s = k} \sum_{q=1}^s \sigma_{i_q}(0, 1)$$

on the annulus, for $s = 2 + s_1 + s_2$.

We now assume that $R_\alpha \rightarrow 1$ as $\alpha \rightarrow +\infty$. Then thanks to the application

$$f(z) = \exp \left(\left(z + \frac{\pi}{4} \right) \frac{2 \ln(R_\alpha)}{\pi} \right),$$

we have

$$f(T_\alpha) = \mathbb{D}_{R_\alpha} \setminus \mathbb{D}$$

with

$$T_\alpha = \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \times [0, b_\alpha] \text{ and } b_\alpha = \frac{\pi^2}{\ln(R_\alpha)} \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty.$$

Notice that we identify $\{Im(z) = 0\}$ and $\{Im(z) = b_\alpha\}$ and that $\{Re(z) = -\frac{\pi}{4}\}$ and $\{Re(z) = \frac{\pi}{4}\}$ correspond to the boundary components of the annulus. We denote by

$$I_\alpha = \left(\left\{ -\frac{\pi}{4} \right\} \cup \left\{ \frac{\pi}{4} \right\} \right) \times [0, b_\alpha]$$

and for $0 \leq r \leq s \leq b_\alpha$,

$$I_\alpha(r, s) = \{(x, y) \in I_\alpha; r \leq y \leq s\}.$$

For sequences $\{r_\alpha\}$ and $\{s_\alpha\}$, $r_\alpha \ll s_\alpha$ means $s_\alpha - r_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. Then, denoting again \tilde{g}_α on T_α the metric $f^*(\tilde{g}_\alpha)$ we claim that

Claim 54. *If some sequences $\{r_\alpha^i\}$ and $\{s_\alpha^i\}$ for $1 \leq i \leq t$ satisfy*

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \dots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

and

$$m_j = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^i, s_\alpha^i)) > 0$$

for $1 \leq i \leq t$, then $t \leq k$.

Proof

We proceed by contradiction and assume that we have such sequences with $t \geq k + 1$. Let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(r_\alpha^{i+1} - s_\alpha^i)$ as $\alpha \rightarrow +\infty$ for $0 \leq i \leq t$. We set for $1 \leq i \leq t$

$$\eta_\alpha^i = \begin{cases} 1 & r_\alpha^i \leq y \leq s_\alpha^i \\ \frac{y - r_\alpha^i + \theta_\alpha}{\theta_\alpha} & r_\alpha^i - \theta_\alpha \leq y \leq r_\alpha^i \\ \frac{s_\alpha^i + \theta_\alpha - y}{\theta_\alpha} & s_\alpha^i \leq y \leq s_\alpha^i + \theta_\alpha \\ 0 & y \geq s_\alpha^i + \theta_\alpha \text{ or } y \leq r_\alpha^i - \theta_\alpha \end{cases}$$

Then,

$$\begin{aligned} \int_{T_\alpha} |\nabla \eta_\alpha^i|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha} &= \int_{T_\alpha} |\nabla \eta_\alpha^i|^2 dx = \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \rightarrow +\infty, \\ \int_{I_\alpha} (\eta_\alpha^i)^2 d\sigma_{\tilde{g}_\alpha} &\geq m_j + o(1) \text{ as } \alpha \rightarrow +\infty. \end{aligned}$$

Taking these at least $k + 1$ functions with pairwise disjoint support for the variational characterization of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$ (6.4) gives that

$$\sigma_\alpha \leq \max_{1 \leq i \leq k+1} \frac{\int_{T_\alpha} |\nabla \eta_i^\alpha|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_{I_\alpha} (\eta_i^\alpha)^2 d\sigma_{\tilde{g}_\alpha}} = o(1) \text{ as } \alpha \rightarrow +\infty$$

which contradicts (6.126). ◇

Now, we prove that up to a rotation on M , there exist sequences $0 \ll r_\alpha \ll s_\alpha \ll b_\alpha$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha, s_\alpha)) = 1. \quad (6.131)$$

Indeed, denying (6.131) would mean that for any sequence $1 \ll u_\alpha \ll v_\alpha \ll b_\alpha$,

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(u_\alpha, v_\alpha)) > 0.$$

Taking for $1 \leq j \leq k + 1$ $y_\alpha^j = \frac{j}{k+2}b_\alpha$ and $\theta_\alpha = \sqrt{b_\alpha}$ gives for $1 \leq j \leq k + 1$

$$m_j = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(y_\alpha^j - \theta_\alpha, y_\alpha^j + \theta_\alpha)) > 0$$

so that the $k + 1$ test functions for $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$ with pairwise disjoint support,

$$\eta_\alpha^j = \begin{cases} 1 & y_\alpha^j - \theta_\alpha \leq y \leq y_\alpha^j + \theta_\alpha \\ \frac{y - y_\alpha^j + 2\theta_\alpha}{\theta_\alpha} & y_\alpha^j - 2\theta_\alpha \leq y \leq y_\alpha^j - \theta_\alpha \\ \frac{y_\alpha^j + 2\theta_\alpha - y}{\theta_\alpha} & y_\alpha^j + \theta_\alpha \leq y \leq y_\alpha^j + 2\theta_\alpha \\ 0 & y \geq y_\alpha^j + 2\theta_\alpha \text{ or } y \leq y_\alpha^j - 2\theta_\alpha \end{cases}$$

would satisfy

$$\begin{aligned} \int_{T_\alpha} |\nabla \eta_j^\alpha|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha} &= \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \rightarrow +\infty, \\ \int_{I_\alpha} (\eta_j^\alpha)^2 d\sigma_{\tilde{g}_\alpha} &\geq m_j + o(1) \text{ as } \alpha \rightarrow +\infty, \end{aligned}$$

so that $\sigma_\alpha = o(1)$ by (6.4). This contradicts again (6.126).

We take a rotation of M so that (6.131) holds. Then, by Claim 54, we can take t the maximal integer such that there exist sequences

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \dots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

with

$$m_j = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j, s_\alpha^j)) > 0$$

and

$$\sum_{j=1}^t m_j = 1 .$$

We define a sequence $r_\alpha^j < y_\alpha^j < s_\alpha^j$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j, y_\alpha^j)) = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(y_\alpha^j, s_\alpha^j)) = \frac{m_j}{2}$$

and

$$\Psi_\alpha^j(x + iy) = \tan(x + i(y - y_\alpha^j))$$

for $z = x + iy \in T_\alpha$. We consider the harmonic map $\check{\Phi}_\alpha^j = \Phi_\alpha \circ (\Psi_\alpha^j)^{-1}$ on \mathbb{D} . We let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(r_\alpha^{j+1} - s_\alpha^j)$ for all $0 \leq j \leq t$. Then,

$$D_\alpha^j = \Psi_\alpha^j(T_\alpha(r_\alpha^j - \theta_\alpha, r_\alpha^j + \theta_\alpha))$$

exhausts \mathbb{D} ,

$$S_\alpha^j = \Psi_\alpha^j(I_\alpha(r_\alpha^j - \theta_\alpha, r_\alpha^j + \theta_\alpha))$$

exhausts S^1 , and

$$\lim_{\alpha \rightarrow +\infty} L_{\check{g}_\alpha}(S_\alpha^j) = m_j ,$$

where $\check{g}_\alpha = (\Psi_\alpha^j)_* \tilde{g}_\alpha$.

Then, we apply Proposition 9 on (\mathbb{D}, ξ) to $\check{\Phi}_\alpha^j : (D_\alpha^j, S_\alpha^j) \rightarrow (\mathbb{B}^{n+1}, S^n)$. In order to define suitable test functions which naturally extend to the surface, we have to prove that $\mathbf{1}_{S_\alpha^j} \check{\Phi}_\alpha^j \cdot \partial_\nu \check{\Phi}_\alpha^j d\theta$ does not concentrate at the poles $(0, 1)$ and $(0, -1)$. Let's prove it by contradiction : if for instance we have

$$\mathbf{1}_{S_\alpha^j} \check{\Phi}_\alpha^j \cdot \partial_\nu \check{\Phi}_\alpha^j d\theta \rightharpoonup_* m\delta_{(0,1)} + \nu \text{ on } S^1$$

with $m > 0$, and $\nu(\{(0, 1)\}) = 0$, then, $\int_{S^1} d\nu > 0$ and up to the extraction of a subsequence, we can build $c_\alpha^j \ll y_\alpha^j$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j - \theta_\alpha, c_\alpha^j)) = m ,$$

so that if we set $\bar{r}_\alpha = y_\alpha^j + \tau_\alpha$ and $\bar{s}_\alpha = c_\alpha^j + \tau_\alpha$ with $\tau_\alpha = \sqrt{y_\alpha^j - c_\alpha^j}$, we have

$$m_j^1 = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j - \theta_\alpha, \bar{s}_\alpha)) > 0 \text{ and}$$

$$m_j^2 = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha(\bar{r}_\alpha, s_\alpha^j + \theta_\alpha)) > 0$$

with $m_j^1 + m_j^2 = m_j$ and this contradicts the maximality of t .

Therefore, we use eigenfunctions associated to the densities associated to the limits of $\check{\Phi}_\alpha^j$ given by Proposition 9 and we follow the computations of Section 6.7.2. This defines test functions for the variational characterization (6.4) of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$. Since (6.2) is strict, as already said,

$$\sigma_k(0, 2) > \max_{i_1 + \dots + i_s = k} \sum_{q=1}^s \sigma_{i_q}(0, 1) ,$$

and we have at least $k + 1$ test functions which would give a contradiction.

6.8.2 The hyperbolic case

Now, we assume that $(\gamma, m) \neq (0, 2)$. We let $\gamma_\alpha^1, \dots, \gamma_\alpha^s$ the geodesics whose length $l_\alpha^1, \dots, l_\alpha^s$ go to 0 as $\alpha \rightarrow +\infty$, where $1 \leq s \leq 3\gamma - 3 + m$ ([55], IV, lemma 4.1) satisfying one of these conditions

- (i) For $1 \leq i \leq s_1$, γ_α^i is a boundary component, that is a closed geodesic such that $\gamma_\alpha^i \subset \partial M$.
- (ii) For $s_1 + 1 \leq i \leq s_2$, γ_α^i is a closed geodesic such that $\gamma_\alpha^i \cap \partial M = \emptyset$.
- (iii) For $s_1 + s_2 + 1 \leq i \leq s_1 + s_2 + s_3 = s$, γ_α^i is a geodesic which crosses two distinct boundary components at its ends.

The collar lemma ([115], lemma 4.2) gives for $1 \leq i \leq s$ an open neighbourhood P_α^i of γ_α^i isometric to the cylinder

$$\{(t, \theta), -\mu_\alpha^i < t < \mu_\alpha^i, 0 \leq \theta \leq 2\pi\}$$

if γ_α^i satisfies (ii) or (iii) and

$$\{(t, \theta), 0 \leq t < \mu_\alpha^i, 0 \leq \theta \leq 2\pi\}$$

if γ_α^i satisfies (i), endowed with the metric

$$\left(\frac{l_\alpha^i}{2\pi \cos\left(\frac{l_\alpha^i t}{2\pi}\right)} \right)^2 (dt^2 + d\theta^2)$$

with

$$\mu_\alpha^i = \frac{\pi}{l_\alpha^i} \left(\pi - 2 \arctan \left(\sinh \left(\frac{l_\alpha^i}{2} \right) \right) \right).$$

Note that the geodesic γ_α^i corresponds to the line $\{t = 0\}$. Note also that in the cases (i) and (ii) we identify the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ and that in the case (iii), the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ correspond to portions of the boundary components crossed by γ_α^i . In the following, we identify P_α^i with the corresponding cylinder.

We denote $M_\alpha^1, \dots, M_\alpha^r$ the connected components of $M \setminus \bigcup_{i=1}^s P_\alpha^i$ so that

$$M = \left(\bigcup_{i=1}^s P_\alpha^i \right) \cup \left(\bigcup_{j=1}^r M_\alpha^j \right)$$

is a disjoint union. For $s_1 + s_2 + 1 \leq i \leq s$, and $-\mu_\alpha^i < a < b < \mu_\alpha^i$, we denote

$$P_\alpha^i(a, b) = \{(t, \theta); a < t < b\}$$

and for $c = \{c^{i,-}, c^{i,+}\}_{s_1+s_2+1 \leq i \leq s}$, we denote $M_\alpha^j(c)$ the connected component of

$$M \setminus \left(\bigcup_{i=1+s_1+s_2}^s P_\alpha^i(-\mu_\alpha^i + c^{i,-}, \mu_\alpha^i - c^{i,+}) \cup \bigcup_{i=s_1+1}^{s_2} \gamma_\alpha^i \right)$$

which contains M_α^j . We also denote

$$I_\alpha^i = M_\alpha^i \cap \partial M$$

and for $c = \{c^{i,-}, c^{i,+}\}_{s_1+s_2+1 \leq i \leq s}$,

$$I_\alpha^i(c) = M_\alpha^i(c) \cap \partial M$$

For all the proof, we identify \mathbb{R}^2 and \mathbb{C} thanks to the map $F(x, y) = x + iy$.

Let $1 \leq i \leq s_1$. Then, γ_α^i satisfies the condition (i). Then, the image by the map $E : z \mapsto e^{iz}$, of P_α^i is an annulus $\mathbb{D} \setminus \mathbb{D}_{e^{-\mu_\alpha^i}}$ which exhausts \mathbb{D} , where S^1 is the image of the closed geodesic. The map $\check{\Phi}_\alpha^i = \Phi_\alpha^i \circ E^{-1} : \mathbb{D} \setminus \mathbb{D}_{e^{-\mu_\alpha^i}} \rightarrow \mathbb{B}^{n+1}$ satisfies the hypotheses of Proposition 9 and we get some regular limits $\check{\Phi}^i, \omega_1^i, \dots, \omega_{t_i}^i$ such that

$$\int_{\mathbb{D}} |\nabla \check{\Phi}^i|^2 dx + \sum_{j=1}^{t_i} \int_{\mathbb{D}} |\nabla \omega_j^i|^2 dx = \lim_{\alpha \rightarrow +\infty} \int_{\gamma_\alpha^i} e^{\mu_\alpha} d\sigma_{\tilde{g}_\alpha}$$

and the conclusion of the proposition holds for some associated scales and gives natural test functions.

Let $s_1 + s_2 + 1 \leq i \leq s$. Then, γ_α^i satisfies the condition (iii). We denote by

$$\Gamma_\alpha^i = \{(\theta, t) \in P_\alpha^i; \theta = 0 \text{ or } \theta = 2\pi\}$$

and for $-\mu_\alpha^i \leq a \leq b \leq \mu_\alpha^i$,

$$\Gamma_\alpha^i(a, b) = \{(\theta, t) \in \Gamma_\alpha^i; a \leq t \leq b\}.$$

We denote $a_\alpha \ll b_\alpha$ if two sequences a_α and b_α satisfy $b_\alpha - a_\alpha \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. Then, we claim that

Claim 55. *If for integers $t_i \geq 0$, some sequences $a_\alpha^{i,l}, b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \dots, r\}$ satisfy*

$$\begin{aligned} -\mu_\alpha^i &\ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \dots \\ &\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_i+1} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i \end{aligned}$$

and for $1 \leq i \leq s$, $1 \leq l \leq t_i$, $j \in J$,

$$m_{i,l} = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l}, b_\alpha^{i,l})) > 0$$

$$m_j = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) > 0,$$

then, $\sum_{i=1}^s t_i + |J| \leq k$.

Proof

By contradiction, we assume that there exist such sequences with $\sum_{i=1}^s t_i + |J| \geq k + 1$. Let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $1 \leq i \leq s$ and $0 \leq l \leq t_i$. We set $\eta_\alpha^{i,l}$ be such that $\text{supp}(\eta_\alpha^{i,l}) \subset P_\alpha^i$ and

$$\eta_{\alpha}^{i,l} = \begin{cases} 1 & a_{\alpha}^{i,l} \leq t \leq b_{\alpha}^{i,l} + \theta_{\alpha} \\ \frac{t - a_{\alpha}^{i,l} + \theta_{\alpha}}{\theta_{\alpha}} & a_{\alpha}^{i,l} - \theta_{\alpha} \leq t \leq a_{\alpha}^{i,l} \\ \frac{b_{\alpha}^{i,l} + \theta_{\alpha} - t}{\theta_{\alpha}} & b_{\alpha}^{i,l} \leq t \leq b_{\alpha}^{i,l} + \theta_{\alpha} \\ 0 & t \geq b_{\alpha}^{i,l} + \theta_{\alpha} \text{ or } t \leq a_{\alpha}^{i,l} - \theta_{\alpha} \end{cases}$$

and η_{α}^j such that $\text{supp}(\eta_{\alpha}^j) \subset M_{\alpha}^j(c_{\alpha} + \theta_{\alpha})$ and if $\{t = \mu_{\alpha}^i\}$ is on the boundary of M_{α}^j ,

$$\eta_{\alpha}^j = \begin{cases} 1 & \mu_{\alpha}^i - c_{\alpha}^{i,+} \leq t \leq \mu_{\alpha}^i \\ \frac{t - \mu_{\alpha}^i + c_{\alpha}^{i,+} + \theta_{\alpha}}{\theta_{\alpha}} & \mu_{\alpha}^i - c_{\alpha}^{i,+} - \theta_{\alpha} \leq t \leq \mu_{\alpha}^i - c_{\alpha}^{i,+} \end{cases}$$

and we proceed the same way for the symmetric case $\{t = -\mu_{\alpha}^i\}$ with $c_{\alpha}^{i,-}$. Taking these at least $k+1$ test functions with pairwise disjoint support for the variational characterization (6.4) of $\sigma_{\alpha} = \sigma_k(M, \tilde{g}_{\alpha})$, we get

$$\sigma_{\alpha} \leq \max \left(\max_{\substack{1 \leq i \leq s \\ 1 \leq l \leq t_i}} \frac{\int_M |\nabla \eta_{\alpha}^{i,l}|_{\tilde{g}_{\alpha}}^2 dv_{\tilde{g}_{\alpha}}}{\int_{\partial M} (\eta_{\alpha}^{i,l})^2 d\sigma_{\tilde{g}_{\alpha}}}, \max_{j \in J} \frac{\int_M |\nabla \eta_{\alpha}^j|_{\tilde{g}_{\alpha}}^2 dv_{\tilde{g}_{\alpha}}}{\int_{\partial M} (\eta_{\alpha}^j)^2 d\sigma_{\tilde{g}_{\alpha}}} \right).$$

Then $\sigma_{\alpha} \leq o(1)$ which contradicts (6.126).

◇

We now prove that the set of such sequences such that

$$\sum_{i=1}^s \sum_{l=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1$$

is not empty.

Claim 56. We let I_0 be the set of indices $i \in \{1, \dots, s\}$ such that there exists a sequence $0 \ll c_{\alpha}^i \ll \mu_{\alpha}^i$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_{\alpha}}(\Gamma_{\alpha}^i(-\mu_{\alpha}^i + c_{\alpha}^i, \mu_{\alpha}^i - c_{\alpha}^i)) = 0$$

and $I_1 = \{1, \dots, s\} \setminus I_0$. Then, there exist sequences $c_{\alpha}^{i,\pm} \rightarrow +\infty$ $0 \ll c_{\alpha}^{i,\pm} \ll \mu_{\alpha}^i$ for $1 \leq i \leq s$ and sequences $a_{\alpha}^i, b_{\alpha}^i$ for $i \in I_1$ with

$$-\mu_{\alpha}^i + c_{\alpha}^{i,+} \ll a_{\alpha}^i \ll b_{\alpha}^i \ll \mu_{\alpha}^i - c_{\alpha}^{i,-},$$

such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_{\alpha}}(\Gamma_{\alpha}^i(-\mu_{\alpha}^i + c_{\alpha}^{i,-}, \mu_{\alpha}^i - c_{\alpha}^{i,+})) = 0$$

for $i \in I_0$,

$$\lim_{\alpha \rightarrow +\infty} \sum_{i=1}^s L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(a_\alpha^i, b_\alpha^i)) > 0$$

for $i \in I_1$ and

$$\lim_{\alpha \rightarrow +\infty} \sum_{i \in I_1} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(a_\alpha^i, b_\alpha^i)) + \sum_{j=1}^r L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) = 1.$$

Proof

We proceed by contradiction, assuming the opposite to hold. Then $I_1 \neq \emptyset$ and we set for $i \in I_1$ and $1 \leq j \leq k+1$

$$\begin{aligned} \mu_\alpha^i - c_\alpha^{i,+} &= \mu_\alpha^i - c_\alpha^{i,-} = t_\alpha^{i,j} + \theta_\alpha \\ b_\alpha^j &= -a_\alpha^j = t_\alpha^j - \theta_\alpha \end{aligned}$$

where $t_\alpha^j = \frac{j\mu_\alpha^i}{k+2}$ and $\theta_\alpha \rightarrow +\infty$ satisfies $\theta_\alpha = o(\mu_\alpha^i)$. Then, by assumption,

$$\sum_{i=1}^s \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha} \left(\Gamma_\alpha^i(-t_\alpha^{i,j} - \theta_\alpha, -t_\alpha^{i,j} + \theta_\alpha) \cup \Gamma_\alpha^i(t_\alpha^{i,j} - \theta_\alpha, t_\alpha^{i,j} + \theta_\alpha) \right) > 0$$

for any $1 \leq j \leq k+1$. We now set $\eta_\alpha^{i,j}$ some test functions for the variational characterization of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$ with pairwise disjoint support defined such that $\text{supp}(\eta_\alpha^j) \subset \bigcup_{i \in I_1} P_\alpha^i$, η_α^i is an even function on P_α^i and

$$\eta_\alpha^{i,j} = \begin{cases} 0 & 0 \leq t \leq t_\alpha^{i,j} - 2\theta_\alpha \\ \frac{t - t_\alpha^{i,j} + 2\theta_\alpha}{\theta_\alpha} & t_\alpha^{i,j} - 2\theta_\alpha \leq t \leq t_\alpha^{i,j} - \theta_\alpha \\ 1 & t_\alpha^{i,j} - \theta_\alpha \leq t \leq t_\alpha^{i,j} + \theta_\alpha \\ \frac{t_\alpha^{i,j} + 2\theta_\alpha - t}{\theta_\alpha} & t_\alpha^{i,j} + \theta_\alpha \leq t \leq t_\alpha^{i,j} + 2\theta_\alpha \\ 0 & t_\alpha^{i,j} + 2\theta_\alpha \leq t \leq \mu_\alpha^i \end{cases}$$

With these $k+1$ test functions, we easily prove that $\sigma_\alpha \leq o(1)$ by (6.4), which contradicts (6.126). \diamond

Thanks to Claim 55 and Claim 56 there exist for $1 \leq i \leq s$ some integers $t_i \geq 0$ sequences $a_\alpha^{i,l}, b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \dots, r\}$ satisfying $c_\alpha^{i,\pm} < \mu_\alpha^i$,

$$\begin{aligned} -\mu_\alpha^i &\ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \dots \\ &\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_i+1} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i \end{aligned}$$

and for $1 \leq i \leq s$, $1 \leq l \leq t_i$, $j \in J$,

$$m_{i,l} = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l}, b_\alpha^{i,l})) > 0$$

$$m_j = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) > 0,$$

with

$$\sum_{i=1}^s \sum_{m=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1$$

such that $\sum_{i=1}^s t_i$ is maximal.

For fixed $1 \leq i \leq s$ and $1 \leq l \leq t_i$, we focus on the asymptotic behaviour of the harmonic map Φ_α on the cylinder $P_\alpha^i(a_\alpha^{i,l}, b_\alpha^{i,l})$. We define a sequence $t_\alpha^{i,l}$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l}, t_\alpha^{i,l})) = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(t_\alpha^{i,l}, b_\alpha^{i,l})) = \frac{m_{i,l}}{2}.$$

We set

$$\Psi_\alpha^{i,l}(\theta + it) = \tan\left(\frac{\theta - \pi + i(t - t_\alpha^{i,l})}{4}\right)$$

and we consider the $\frac{1}{2}$ -harmonic map $\check{\Phi}_\alpha^{i,l} = \Phi_\alpha \circ (\Psi_\alpha^{i,l})^{-1}$ on \mathbb{D} . Let $\theta_\alpha \rightarrow +\infty$ be such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $0 \leq l \leq t_i$ and $1 \leq i \leq s$. Then,

$$D_\alpha^{i,l} = \Psi_\alpha^{i,l}\left(T_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, b_\alpha^{i,l} + \theta_\alpha)\right)$$

exhausts \mathbb{D} ,

$$S_\alpha^{i,l} = \Psi_\alpha^{i,l}\left(\Gamma_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, b_\alpha^{i,l} + \theta_\alpha)\right)$$

exhausts \mathbb{S}^1 and

$$\lim_{\alpha \rightarrow +\infty} L_{(\Psi_\alpha^{i,l})_\star(\tilde{g}_\alpha)}(S_\alpha^{i,l}) = m_{i,l}.$$

We can now apply Proposition 9 on (\mathbb{D}, ζ) to $\check{\Phi}_\alpha^{i,l}(D_\alpha^{i,l}, S_\alpha^{i,l}) \rightarrow (\mathbb{B}^{n+1}, \mathbb{S}^n)$. In order to obtain test functions which naturally extend to the manifold, we have to prove that $\mathbf{1}_{S_\alpha^{i,l}} \check{\Phi}_\alpha^{i,l} \partial_\nu \check{\Phi}_\alpha^{i,l} d\theta$ does not concentrate at the poles $(0, 1)$ and $(0, -1)$. By contradiction, if we have

$$\mathbf{1}_{S_\alpha^{i,l}} \check{\Phi}_\alpha^{i,l} \partial_\nu \check{\Phi}_\alpha^{i,l} d\theta \rightharpoonup_\star m\delta_{(0,1)} + \nu$$

with $m > 0$, $\nu(\{(0, 1)\}) = 0$, then $\int_{\mathbb{S}^1} d\nu > 0$ by the hypothesis on $t_\alpha^{i,l}$ we did and up to the extraction of a subsequence, we can build $q_\alpha^{i,l} \ll t_\alpha^{i,l}$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l} - \theta_\alpha, q_\alpha^{i,l})) = m.$$

Setting $\bar{b}_\alpha = q_\alpha^{i,l} + \tau_\alpha$ and $\bar{a}_\alpha = t_\alpha^{i,l} - \tau_\alpha$, with $\tau_\alpha = \sqrt{t_\alpha^{i,l} - q_\alpha^{i,l}}$, we have

$$m_{i,l}^1 = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, \bar{b}_\alpha)) > 0$$

$$m_{i,l}^2 = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(\bar{a}_\alpha, b_\alpha^{i,l} + \theta_\alpha)) > 0$$

with $m_{i,l}^1 + m_{i,l}^2 = m_{i,l}$ and this contradicts the maximality of $\sum_{i=1}^s t_i$.

For a fixed $j \in J$, we now focus on the asymptotic behaviour of Φ_α on $M_\alpha^j(c_\alpha)$. We denote by \widetilde{M}_α^j the connected component of $M \setminus (\gamma_\alpha^1, \dots, \gamma_\alpha^s)$ which contains M_α^j . There exists a diffeomorphism $\tau_\alpha : \Sigma_j \rightarrow \widetilde{M}_\alpha^j$ such that (Σ_j, h_α) is a non compact hyperbolic surface with $h_\alpha = \tau_\alpha^* g_\alpha$. On Σ_j , we have

$$h_\alpha \rightarrow h \text{ in } C_{loc}^\infty(\Sigma_j) \text{ as } \alpha \rightarrow +\infty$$

for a hyperbolic metric h . We let $c = [h]$ and $(\hat{\Sigma}_j, \hat{c})$ the compactification of the cusps of (Σ_j, h) so that $(\hat{\Sigma}_j \setminus \{p_1, \dots, p_t\}, \hat{c})$ is conformal to (Σ_j, c) for some punctures p_1, \dots, p_t as described in [55]. The sequence of sets $\Sigma_\alpha = \tau_\alpha^{-1}(M_\alpha^j(c_\alpha))$ exhausts $\hat{\Sigma}_j$, so that the sequence of harmonic maps with free boundary $\hat{\Phi}_\alpha = \Phi_\alpha \circ \tau_\alpha : (\Sigma_\alpha, h_\alpha) \rightarrow \mathbb{B}^{n+1}$ satisfies the hypotheses of Proposition 9. In order to extend on the whole manifold the suitable test functions we define on Σ_j , we will prove that $\mathbf{1}_{\Sigma_\alpha} \hat{\Phi}_\alpha \cdot \partial_{\nu_\alpha} \hat{\Phi}_\alpha d\sigma_{h_\alpha}$ does not concentrate at the punctures which lie in the boundary of $\hat{\Sigma}_j$ (and correspond to the degeneration of some geodesic γ_α^i which satisfies condition (iii)). By contradiction, we assume that

$$\mathbf{1}_{\Sigma_\alpha} \hat{\Phi}_\alpha \cdot \partial_{\nu_\alpha} \hat{\Phi}_\alpha d\sigma_{h_\alpha} \rightharpoonup_* m\delta_{p_l} + \nu \text{ on } \hat{\Sigma}_j$$

for some puncture $p_l \in \{p_1, \dots, p_t\} \cap \partial\hat{\Sigma}_j$ with $m > 0$, $\nu(\{p_l\}) = 0$. Then, up to the extraction of a subsequence, we can build $q_\alpha \rightarrow +\infty$ such that

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha} \left(\Gamma_\alpha^i(-\mu_\alpha^i + q_\alpha, -\mu_\alpha^i + c_\alpha^{i,-}) \right) = m$$

for $s_1 + s_2 + 1 \leq i \leq s$ such that $\tau_\alpha^{-1}(\{-\mu_\alpha^i < t < 0\})$ is a neighbourhood of the puncture p_l of $\hat{\Sigma}_j$. We proceed the same way for the symmetric case $\{0 < t < \mu_\alpha^i\}$. Setting $d_\alpha = \sqrt{q_\alpha}$, $\bar{a}_\alpha = -\mu_\alpha^i + q_\alpha - \sqrt{q_\alpha}$ and $\bar{b}_\alpha = -\mu_\alpha^i + c_\alpha^{i,-}$, we have

$$m = \lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha} \left(\Gamma_\alpha^i(\bar{a}_\alpha, \bar{b}_\alpha) \right) > 0$$

$$\lim_{\alpha \rightarrow +\infty} L_{\tilde{g}_\alpha} I_\alpha^j(\bar{c}_\alpha) = m_j - m$$

where \bar{c}_α comes from c_α , taking d_α instead of $c_\alpha^{i,-}$. Adding the sequences $\bar{a}_\alpha \ll \bar{b}_\alpha$ contradicts the maximality of $\sum_{i=1}^s t_i$.

As described in Proposition 9 and the computations of section 6.7.2, the limit functions given by $\check{\Phi}_\alpha^i : D_\alpha^i \subset \mathbb{D} \rightarrow \mathbb{B}^{n+1}$ for $1 \leq i \leq s_1$, $\check{\Phi}_\alpha^{i,l} : D_\alpha^{i,l} \subset \mathbb{D} \rightarrow \mathbb{B}^{n+1}$ for $s_1 + s_2 + 1 \leq i \leq s$ and $\hat{\Phi}_\alpha^j : \Sigma_\alpha \subset \hat{\Sigma}_j \rightarrow \mathbb{B}^{n+1}$ and their associated scales give at least $k+1$ well defined test functions for the variational characterization of σ_α by the gap (6.2). Indeed, denoting γ_j the genus of $\hat{\Sigma}_j$ and m_j its number of boundary components, we notice that $\sum_{j \in J} \gamma_j \leq \gamma$ and $\sum_{j \in J} m_j \leq m$ and that if $|J| = 1$, $\gamma_1 < \gamma$ or $m_1 < m$. These at least $k+1$ test functions for the variational characterization (6.4) of σ_α give a contradiction. This ends the proof of Theorem 14.

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Bornes sur des valeurs propres et métriques extrémales

Résumé : Cette thèse est consacrée à l'étude des valeurs propres de l'opérateur de Laplace et de l'opérateur de Steklov sur des variétés riemanniennes. On cherche à donner des bornes optimales parmi l'ensemble des métriques, dans une classe conforme donnée ou non, et à caractériser, si elles existent, les métriques qui atteignent ces bornes. Ces métriques extrémales ont des propriétés qui s'inscrivent dans la théorie des surfaces minimales.

On s'intéresse d'abord à la borne supérieure des valeurs propres de Laplace parmi des métriques conformes entre elles, appelées valeurs propres conformes. Dans le chapitre 1, on estime la deuxième valeur propre conforme de la sphère standard. Dans les chapitres 2 et 3, on montre que la première valeur propre conforme d'une variété riemannienne est plus grande que celle de la sphère standard de même dimension avec égalité seulement pour la sphère standard.

Ensuite, on cherche à démontrer l'existence et la régularité de métriques qui maximisent les valeurs propres sur des surfaces, dans une classe conforme donnée ou non. Dans les chapitres 3 et 4, on démontre un résultat d'existence pour les valeurs propres de Laplace. Dans le chapitre 6, le travail est fait pour les valeurs propres de Steklov.

Enfin, dans le chapitre 5, fruit d'un travail réalisé en collaboration avec Paul Laurain, on démontre un résultat de régularité et de quantification des applications harmoniques à bord libre sur une surface Riemannienne. C'est un élément clé pour le chapitre 6.

Mots clés : valeurs propres de Laplace, valeurs propres de Steklov, valeurs propres conforme, métriques extrémales, surfaces minimales.

Eigenvalue bounds and extremal metrics

Abstract : This thesis is devoted to the study of the Laplace eigenvalues and the Steklov eigenvalues on Riemannian manifolds. We look for optimal bounds among the set of metrics, lying in a conformal class or not. We also characterize, if they exist the metrics which reach these bounds. These extremal metrics have properties from the theory of minimal surfaces.

First, we are interested in the upper bound of Laplace eigenvalues in a class of conformal metrics, called the conformal eigenvalues. In Chapter 1, we estimate the second conformal eigenvalue of the standard sphere. In Chapters 2 and 3, we prove that the first conformal eigenvalue of a Riemannian manifold is greater than the one of the standard sphere of same dimension, with equality only for the standard sphere. Then, we look for existence and regularity results for metrics which maximize eigenvalues on surfaces, in a given conformal class or not. In Chapters 3 and 4, we prove an existence result for Laplace eigenvalues. In Chapter 6, the work is done for Steklov eigenvalues.

Finally, in Chapter 5, obtained in collaboration with Paul Laurain, we prove a regularity and quantification result for harmonic maps with free boundary on a Riemannian surface. It is a key component for Chapter 6.

Keywords : Laplace eigenvalues, Steklov eigenvalues, conformal eigenvalues, extremal metrics, minimal surfaces.

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